

Commutative Algebra

lecture 3: Noetherian rings and irreducible affine varieties

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Irreducible varieties

DEFINITION: An affine variety A is called **reducible** if it can be expressed as a union $A = A_1 \cup A_2$ of affine varieties, such that $A_1 \not\subset A_2$ and $A_2 \not\subset A_1$. If such a decomposition is impossible, A is called **irreducible**.

CLAIM: An affine variety A is **irreducible** if and only if its ring of polynomial functions \mathcal{O}_A **has no zero divisors**.

Proof. Step 1: If $A = A_1 \cup A_2$ is a decomposition of A into a non-trivial union of subvarieties, choose a non-zero function $f \in \mathcal{O}_A$ vanishing at A_1 and g vanishing at A_2 . Such functions exist by Nullstellensatz. Indeed, if all functions which vanish in A_1 vanish in A , Nullstellensatz implies that $A_1 \supset A$. The product of these non-zero functions vanishes in $A = A_1 \cup A_2$, **hence $fg = 0$ in \mathcal{O}_A** .

Step 2: Conversely, **if $fg = 0$, we decompose $A = V_f \cup V_g$** . This decomposition is non-trivial, because $A \subset V_f$ implies that $f = 0$ in \mathcal{O}_A . ■

Noetherian rings and irreducible components

DEFINITION: A ring is called **Noetherian** if any increasing chain of ideals stabilizes: for any chain $I_1 \subset I_2 \subset I_3 \subset \dots$ one has $I_n = I_{n+1} = I_{n+2} = \dots$

THEOREM: (Hilbert basis theorem)

Any finitely generated ring is Noetherian.

Proof: *Later today.*

DEFINITION: An irreducible component of an algebraic set A is an irreducible algebraic subset $A' \subset A$ such that $A = A' \cup A''$, and $A' \not\subset A''$.

Remark 1: Let $A_1 \supset A_2 \supset \dots \supset A_n \supset \dots$ be a decreasing chain of algebraic subsets in an algebraic variety. **Then the corresponding ideals form an increasing chain of ideals:** $\text{Ann}(A_1) \subset \text{Ann}(A_2) \subset \text{Ann}(A_3) \subset \dots$

THEOREM: Let A be an affine variety, and \mathcal{O}_A its ring of polynomial functions. Assume that \mathcal{O}_A is Noetherian. Then **A is a union of its irreducible components, which are finitely many.**

Proof: Later today. ■

Remark 2: From the noetherianity and Remark 1 it follows that A **cannot contain a strictly decreasing infinite chain of algebraic subvarieties.**

Noetherian topological space and Zariski topology

DEFINITION: A topological space is called **Noetherian** if any decreasing chain of closed subvarieties stabilizes.

EXAMPLE: Let A be an algebraic variety. A subset $B \subset A$ is **Zariski closed** if B is algebraic (that is, a zero set of an ideal $I \subset \mathcal{O}_A$). Since an intersection and a finite union of algebraic sets is algebraic, this defines a topology on A , called **the Zariski topology**.

REMARK: By Remark 2, **an algebraic variety with Zariski topology is a Noetherian topological space.**

CLAIM: An algebraic variety **is compact in Zariski topology.**

Proof: An open set of A is a complement to $V(I)$, where I is an ideal. Therefore, a covering of A by open sets $\{U_k\}$ is the same as a collection of ideals $\{I_k\}$ which have no common zeros. By Nullstellensatz, $\sum_k I_k \ni 1$, hence 1 belongs to a finite combination of elements from I_{k_1}, \dots, I_{k_n} , which implies $\bigcup_{i=1}^n U_{k_i} = A$. ■

Irreducible decomposition in Noetherian topological spaces

DEFINITION: A closed subset of a Noetherian topological space is **irreducible** if it is not a union of proper closed subsets. An **irreducible component** of a closed set A is an irreducible subset $B \subset A$ such that $A = B \cup B'$, and $B' \subset A$ is a closed subset not containing B .

CLAIM: Any point x in a Noetherian topological space A **is contained in an irreducible component** $B \subset A$.

Proof: Suppose that x does not belong to an irreducible component. Then for each non-trivial decomposition $A = A_1 \cup A_2$, with $A_1 \ni x$, the component A_1 can be non-trivially split onto two more components, say, A_{11} and A_{12} , with $x \in A_{11}$, and A_{11} can be split, $A_{11} = A_{111} \cup A_{112}$. and so on. **This produces an infinite strictly decreasing chain** $A \supset A_1 \supsetneq A_{11} \supsetneq \dots$, which contradicts Noetherianity. We obtained that **A is a union of its irreducible components.** ■

Irreducible decomposition in Noetherian topological spaces (2)

PROPOSITION: Any Noetherian topological space A **can be obtained as a finite union of its irreducible components.**

Proof: By the previous claim, any point of A belongs to an irreducible component. Therefore, unless A is irreducible (and we are done), **we have $A = A_1 \cup A_2$, where A_1 is irreducible and $A_2 \subsetneq A$ is closed.** If A_2 is a union of finitely many irreducible components, we are done. Otherwise we will decompose A_2 onto a union of A_{12} (which is irreducible) and A_{22} , which is strictly smaller than A_2 . This process terminates, because the sequence $A \supset A_2 \supset A_{22} \supset \dots$ terminates by Noetherianity. ■

THEOREM: Let A be an affine variety, and \mathcal{O}_A its ring of polynomial functions. Assume that \mathcal{O}_A is Noetherian. Then **A is a union of its irreducible components, which are finitely many.**

Proof: Indeed, A is Noetherian in the Zariski topology. ■

Noetherian rings

DEFINITION: A finitely generated ring is a quotient of a polynomial ring.

THEOREM: (Hilbert's Basis Theorem)

Any finitely generated ring over a field is Noetherian.

Proof: Later in this lecture.

COROLLARY: For any affine variety, its ring of functions is Noetherian, hence the irreducible decomposition exists and is finite.

REMARK: It suffices to prove Hilbert's Basis Theorem for the ring of polynomials. Indeed, any finitely generated ring is a quotient of the polynomial ring, hence the set of ideals of the quotient ring A/I is injectively mapped to the set of ideals of R .

REMARK: Therefore, Hilbert's Basis Theorem would follow if we prove that $R[t]$ is Noetherian for any Noetherian ring R .

EXERCISE: Find an example of a ring which is not Noetherian.

Emmy Noether



Amalie Emmy Noether (1882-1935).

Emanuel Lasker



Emanuel Lasker (1868-1941).

Finitely generated ideals

DEFINITION: **Finitely generated ideal** in a ring R is an ideal $\langle a_1, \dots, a_n \rangle$ consisting of all sums $\sum b_i a_i$, where $b_i \in R$. Here $\{a_i\}$ is a fixed finite set of elements of R , called **the generators** of R .

LEMMA: Let $I \subset R$ be a finitely generated ideal, and $I_0 \subset I_1 \subset I_2 \subset \dots$ an increasing chain of ideals, such that $\bigcup_n I_n = I$. **Then this chain stabilises.**

Proof: Let $I = \langle a_1, \dots, a_n \rangle$, and I_N be an ideal in the chain $I_0 \subset I_1 \subset I_2 \subset \dots$ which contains all a_i . Then $I_N = I$. ■

CLAIM: **A ring R is Noetherian if and only if all its ideals are finitely generated.**

Proof. Step 1: For any chain of ideals $I_0 \subset I_1 \subset I_2 \subset \dots$, **finite generatedness of $I = \bigcup I_i$ guarantees stabilization of this chain**, as follows from Lemma above.

Step 2: Conversely, if R is Noetherian, and I any ideal, take $I_0 = 0$ and let $I_k \subset I$ be obtained by adding to I_{k-1} an element of I not contained in I_{k-1} . **Since the chain $\{I_k\}$ stabilizes, I is finitely generated.** ■

Noetherian modules

DEFINITION: A module over a ring R is a vector space M equipped with an algebra homomorphism $R \rightarrow \text{End}(M)$.

EXAMPLE: A subspace $I \subset R$ in a ring is an ideal if and only if I is an R -submodule of R , considered as an R -module.

DEFINITION: A module M over R is called **Noetherian** if any increasing chain of submodules of M stabilizes.

REMARK: Any submodules and quotient modules of a Noetherian R -module are again Noetherian.

Finitely generated R -modules

DEFINITION: An R -module is called **finitely generated** if it is a quotient of a **free module** R^n by its submodule.

EXERCISE: Show that **a module M is Noetherian iff any $M' \subset M$ is finitely generated.** Use this to prove that **finite direct sums of Noetherian modules are Noetherian.**

LEMMA: A ring R is Noetherian if and only if it is Noetherian as an R -module.

Proof: Ideals in R is the same as R -submodules of R , stabilization of a chain of R -submodules in R is literally the same as stabilization of a chain of ideals in R . ■

REMARK: Let M be a module over $R[t]$ which is Noetherian as an R -module, **Then it is Noetherian as $R[t]$ -module.** ■

COROLLARY: If R is Noetherian, then $R[t]/(t^N) = R^N$ is a Noetherian R -module. Therefore, **the ring $R[t]/(t^N)$ is Noetherian.** ■

An ideal generated by leading terms

Proposition 1: Let $J \subset R[t]$ be an ideal, and $Q_1(t), \dots, Q_n(t) \in J$ be polynomials of degree k . Let $J_0 \subset R$ be an ideal generated by the leading terms of $Q_i(t)$. Suppose that all leading terms of all $P(t) \in J$ belong to J_0 . Then, **for each $P(t) \in J$, there is an element $P_1(t) \in J$, $\deg P_1(t) < k$, such that $P(t) = P_1(t) \pmod{J_Q}$** , where $J_Q = (Q_1(t), \dots, Q_n(t))$ is the ideal generated by all $Q_i(t)$.

Proof. Step 1: Let $P(t), Q(t)$ be polynomials in $R[t]$, $n = \deg P > k = \deg Q$, p, q their leading coefficients. Suppose that $p = q$. **Then $P(t) = Q(t)t^{n-m} + P_1(t)$, where $\deg P_1 < \deg P$.**

Step 2: For each $P(t) \in J$ of degree $n > k$, there exists a polynomial $Q(t) = t^{n-k} \sum u_i Q_i(t) \in J_Q$, $u_i \in R$, with the same degree and the leading term. **Then $\deg(P(t) - Q(t)) < \deg P(t)$.** Using induction by $\deg P$, we may assume that $P(t) - Q(t) = P_1(t) \pmod{J_Q}$, where $\deg P_1(t) < k$, hence $P(t) = P_1(t) \pmod{J_Q}$. ■

Proof of Hilbert's basis theorem

PROPOSITION: Let R be a Noetherian ring. **Then the polynomial ring $R[t]$ is also Noetherian.**

Proof. Step 1: Let $I \subset R[t]$ be an ideal. We need to show that it is finitely generated. Consider the ideal $I_0 \subset R$ generated by all leading coefficients of all $P(t) \in I$. Since R is Noetherian, I_0 is finitely generated: $I_0 = \langle a_1, \dots, a_n \rangle$, where all a_i are leading coefficients of the finite set of polynomials $Q_i(t) \in I$.

Step 2: Let N be the maximum of all degrees of $Q_i(t)$, and $I_Q \subset R[t]$ the ideal generated by all $Q_i(t)$. By Proposition 1, for any $P(t) \in I$, one has $P(t) = P_1(t) \pmod{I_Q}$, where $\deg P_1(t) < N$.

Step 3: We have shown that the natural map $I \cap R\langle 1, t, t^2, \dots, t^{N-1} \rangle \longrightarrow I/I_Q$ is surjective. However, $I \cap R\langle 1, t, t^2, \dots, t^{N-1} \rangle \subset R^N$ is Noetherian as an R -module, hence $M := I/I_Q$ is finitely generated as an R -module.

Step 4: Pick a set of polynomials $U_1(t), \dots, U_m(t) \in I$, generating M . **Then $\{Q_i(t), U_i(t)\}$ generate I . ■**