Commutative Algebra

lecture 4: Irreducible varieties: smoothness and primary decomposition

Misha Verbitsky

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Irreducible varieties (reminder)

DEFINITION: An affine variety A is called **reducible** if it can be expressed as a union $A = A_1 \cup A_2$ of affine varieties, such that $A_1 \not\subset A_2$ and $A_2 \not\subset A_1$. If such a decomposition is impossible, A is called **irreducible**.

CLAIM: An affine variety A is **irreducible** if and only if its ring of polynomial functions $\mathcal{O}_A$ **has no zero divizors.**

Proof. Step 1: If $A = A_1 \cup A_2$ is a decomposition of A into a non-trivial union of subvarieties, choose a non-zero function $f \in \mathcal{O}_A$ vanishing at A_1 and g vanishing at A_2. Such functions exist by Nullstellensatz. Indeed, if all functions which vanish in A_1 vanish in A, Nullstellensatz implies that $A_1 \supset A$. The product of these non-zero functions vanishes in $A = A_1 \cup A_2$, **hence $fg = 0$ in $\mathcal{O}_A$.**

Step 2: Conversely, **if $fg = 0$, we decompose $A = V_f \cup V_g$.** This decomposition is non-trivial, because $A \subset V_f$ implies that $f = 0$ in $\mathcal{O}_A$. ■

DEFINITION: An irreducible component of an algebraic set A is an irreducible algebraic subset $A' \subset A$ such that $A = A' \cup A''$, and $A' \not\subset A''$.

THEOREM: Let A be an affine variety, and $\mathcal{O}_A$ its ring of polynomial functions. Then **A is a union of its irreducible components, which are finitely many.**

Noetherian rings (reminder)

DEFINITION: (Claude Chevalley)
A ring is called **Noetherian** if any increasing chain of ideals stabilizes: for any chain $I_1 \subset I_2 \subset I_3 \subset ...$ one has $I_n = I_{n+1} = I_{n+2} = ...$

REMARK: In this case we also say that the chain $I_1 \subset I_2 \subset I_3 \subset ...$ **terminates**.

DEFINITION: A finitely generated ring over $\mathbb{C}$ is a quotient of a polynomial ring $\mathbb{C}[t_1, ..., t_n]$ by an ideal.

THEOREM: (Hilbert's Basis Theorem)
Any finitely generated ring over a field is Noetherian.
Proof: Lecture 3.

COROLLARY: For any affine variety, **its ring of functions is Noetherian.**
■

REMARK: This is used to show that **any algebraic variety admits an irreducible decomposition.**

Smooth points

DEFINITION: Let $A \subset \mathbb{C}^n$ is an algebraic subset. A point $a \in A$ is called **smooth**, if there exists a neighbourhood U of $a \in \mathbb{C}^n$ (in the usual topology) such that $A \cap U$ is a smooth $2k$-dimensional real submanifold. A point is called **singular** if such diffeomorphism does not exist. A variety is called **smooth** if it has no singularities, and **singular** otherwise.

PROPOSITION: For any algebraic variety A and any smooth point $a \in A$, a diffeomorphism between a neighbourhood of a and an open ball **can be chosen polynomial**.

Proof. Step 1: If $a \in A \subset \mathbb{C}^n$ is a smooth point of a k-dimensional embedded manifold, **there exists k complex linear functions on $\mathbb{C}^n$ such that their differentials are linearly independent in the tangent space $T_a A$.**

Step 2: By inverse function theorem, these functions **define a diffeomorphism from a neighbourhood of A to an open subset of $\mathbb{C}^k$.** ■

Analytic functions

DEFINITION: A smooth function complex-valued f on an open subset $U \subset \mathbb{C}^n$ is called **complex analytic** if for any $x \in U$ there is a neighbourhood $U' \ni x$ such that the Taylor series of f, computed in x, converge to f in U'.

REMARK: In the coordinates defined by linear functions **all regular (polynomial) functions on a complex submanifold $A \subset \mathbb{C}^n$ are analytic**, because the polynomials are clearly analytic.

In other polynomial coordinate systems, the Taylor series may be no longer finite, but **the regular functions remain analytic,** because the inverse function theorem remains true in the complex analytic category.

REMARK: In fact, **all complex differentiable functions are complex analytic** (Cauchy), and the regular functions are clearly complex differentiable.

Irreducibility for smooth varieties

CLAIM: Let M be an algebraic variety which is smooth and connected. **Then it is irreducible.**

Proof: ("Analytic continuation principle"). Step 1
Let f, g be non-zero polynomial functions, $fg = 0$. Decomposing f, g onto Taylor series around $m \in M$, **we obtain that the Taylor series for f or for g vanish.** Suppose it is f which has vanishing Taylor series, and let $U \subset M$ be the set where all derivatives of f vanish.

Step 2: Since U is an intersection of closed sets $\{x \in M \mid f^{(i)}(x) = 0\}$, it is closed. However, an analytic function which has vanishing Taylor series in x has to vanish in a neighbourhood of x, hence U is also open. **An open and closed subset of M is M or $\emptyset$, because M is connected.** ■

Irreducibility for smooth varieties (2)

COROLLARY: Let A be an affine variety such that its set A_0 of smooth points is dense in A and connected. **Then A is irreducible.**

Proof: If f and g are non-zero function such that $fg = 0$, the ring of polynomial functions on A_0 contains zero divizors. However, **on a smooth, connected complex manifold the ring of polynomial functions has no zero divisors by analytic continuation principle.** ■

REMARK: Converse is also true: **an algebraic variety over $\mathbb{C}$ is irreducible if and only if the set of its smooth points is connected.** This is a complicated result.

EXERCISE: Let $X \longrightarrow Y$ be a morphism of affine manifols, where X is irreducible, and its image in Y is dense. **Prove that Y is also irreducible.**

Radical ideals

DEFINITION: Let $\mathfrak{u} \subset R$ be an ideal. A **radical** of $\mathfrak{u}$ is the ideal

$$\sqrt{\mathfrak{u}} := \{x \in R \mid x^n \in \mathfrak{u} \text{ for some } n \in \mathbb{Z}^{>0}.\}$$

An ideal $\mathfrak{u} \subset R$ is called **radical** if $\mathfrak{u} = \sqrt{\mathfrak{u}}$

REMARK: An ideal $\mathfrak{u} \subset R$ is radical if and only if $R/\mathfrak{u}$ **has no non-zero nilpotents**.

REMARK: Radical ideals in a finitely-generated ring $R = \mathcal{O}_A$ **are in bijective correspondence with algebraic subsets of A** (this is one of the forms of strong Hilbert Nullstellensatz).

CLAIM: An ideal $\mathfrak{u} \subset R$ is radical **if and only if it is an intersection of prime ideals.**

Proof. Step 1: Prime ideals containing $\mathfrak{u}$ are the same as prime ideals in $R/\mathfrak{u}$. Therefore, it suffices to prove that **0 is a radical ideal if and only if it is an intersection of prime ideals.**

Step 2: The quotient $R/\mathfrak{u}$ **has no nilpotents if and only if** 0 **is intersection of all prime ideals in $R/\mathfrak{u}$** (Lecture 2). ■

Primary ideals

DEFINITION: (Emanuel Lasker)
An ideal $\mathfrak{u}$ is called **primary** if $\sqrt{\mathfrak{u}}$ is a prime ideal.

PROPOSITION: Let $\mathfrak{u} \subset R = \mathcal{O}_A$ be an ideal, and $V_\mathfrak{u} \subset A$ its zero set. Then **$\mathfrak{u}$ is primary if and only if the algebraic variety $V_\mathfrak{u}$ is irreducible.**

Proof. Step 1: By Hilbert Nullstellensatz, the annihilator ideal $\operatorname{Ann}_{V_\mathfrak{u}}$ coincides with $\sqrt{\mathfrak{u}}$. Indeed, $V_\mathfrak{u} = V_{\sqrt{\mathfrak{u}}}$, hence, by Hilbert Nullstellensatz, $\operatorname{Ann}_{V_{\sqrt{\mathfrak{u}}}} = \sqrt{\mathfrak{u}}$. However, $V_{\sqrt{\mathfrak{u}}} = V_\mathfrak{u}$, because $f(x) = 0 \Leftrightarrow f^n(f) = 0$. This gives

$$\operatorname{Ann}_{V_\mathfrak{u}} = \operatorname{Ann}_{V_{\sqrt{\mathfrak{u}}}} = \sqrt{\mathfrak{u}}.$$

Step 2: The variety $A = V_{\sqrt{\mathfrak{u}}}$ is irreducible if and only if $\mathcal{O}_A = \frac{\mathbb{C}[t_1,...,t_n]}{\operatorname{Ann}_A}$ has no zero divisors, which is equivalent to $\operatorname{Ann}_A = \sqrt{\mathfrak{u}}$ being a prime ideal. ■

Primary ideals (2)

CLAIM: An ideal $\mathfrak{u}$ is primary if and only if for any $x, y \in R$ such that $xy \in \mathfrak{u}$, **one has $x^n \in \mathfrak{u}$ or $y^n \in \mathfrak{u}$, for n sufficiently big.**

Proof. Step 1: Suppose that for any $x, y \in R$ such that $xy \in \mathfrak{u}$, one has $x^n \in \mathfrak{u}$ or $y^n \in \mathfrak{u}$ for n sufficiently big. $x^n \in \mathfrak{u}$ for n sufficiently big is equivalent to $x \in \sqrt{\mathfrak{u}}$. By definition of $\sqrt{\mathfrak{u}}$, **for any x, y such that $xy \in \sqrt{\mathfrak{u}}$, one has $x^m y^m \in \mathfrak{u}$ for $m \gg 0$, which implies that $x^{mn} \in \mathfrak{u}$ or $y^{mn} \in \mathfrak{u}$ for $n \gg 0$.** Then $xy \in \sqrt{\mathfrak{u}}$ implies that $x \in \sqrt{\mathfrak{u}}$ or $y \in \sqrt{\mathfrak{u}}$, hence $\sqrt{\mathfrak{u}}$ is prime.

Step 2: Conversely, assume that $\sqrt{\mathfrak{u}}$ is prime. Then for any x, y such that $xy \in \mathfrak{u} \subset \sqrt{\mathfrak{u}}$, one has $x \in \sqrt{\mathfrak{u}}$ or $y \in \sqrt{\mathfrak{u}}$, because $\mathfrak{u}$ is prime; **this implies that $x^m \in \sqrt{\mathfrak{u}}$ or $y^m \in \sqrt{\mathfrak{u}}$.** ■

Irreducible ideals

DEFINITION: Suppose that J, and J_i are ideals in R, and J is represented as $J = \bigcap_i J_i$. The decomposition $J = \bigcap_i J_i$ is called **non-trivial** if $J_i \neq J$ and $J_i \not\subset J_j$ for all i, j. An ideal $J \subset R$ is called **irreducible** if it does not admit a non-trivial decomposition $J = \bigcap_i J_i$. **An irreducible decomposition** of J is a non-trivial decomposition $J = \bigcap_i J_i$, where all J_i are irreducible.

LEMMA: In a Noetherian ring R, **every ideal admits an irreducible decomposition**.

Proof: Let $\mathfrak{R}$ be the set of all ideals not admitting an irreducible decomposition, ordered by inclusion, and J a maximal element in this set; it exists, because R is Noetherian, unless $\mathfrak{R}$ is empty. Since J is not irreducible, we can decompose J as $\bigcap_i J_i$, where all J_i are strictly bigger than J, hence admit an irreducible decomposition $J_i = \bigcap_j J_{ij}$. Then $J = \bigcap_{i,j} J_{ij}$ gives an irreducible decomposition for J. ■

Primary vs. irreducible

LEMMA: An irreducible ideal $J \subset R$ in a Noetherian ring is primary.

Proof. Step 1: Replacing R by R/J, we find that it suffices to show that 0 is primary when it is irreducible. Let $xy = 0$ be some non-trivial zero divisors in R, and $\mathfrak{A}(x^k) := \{z \in R \ \ |zx^k = 0\}$. Since the chain $\mathfrak{A}(x) \subset \mathfrak{A}(x^2) \subset ...$ stabilizes, we have $\mathfrak{A}(x^n) = \mathfrak{A}(x^{n+1})$ for some $n > 0$.

Step 2: The ideals (x^n) and (y) generated by x^n and y satisfy $(x^n) \cap (y) = 0$. Indeed, each $a \in (x^n) \cap (y)$ satisfies $a \in \mathfrak{A}(x) \cap (x^n)$, hence $a = bx^n$ and $bx^{n+1} = 0$, giving $b \in \mathfrak{A}(x^{n+1})$. Since $\mathfrak{A}(x^n) = \mathfrak{A}(x^{n+1})$, this implies $a = bx^n = 0$. Since 0 is irreducible, this implies $x^n = 0$, hence 0 is primary (all zero divisors are nilpotents). ■

REMARK: Primary **does not necessarily imply irreducible.** Indeed, let $\langle x^2, y^2 \rangle \subset \mathbb{C}[x,y]$ be the ideal generated by x^2 and y^2. Then $\langle x^2, y^2 \rangle = \langle x, y^2 \rangle \cap \langle x^2, y \rangle$, which is a non-trivial decomposition. However, $\langle x^2, y^2 \rangle$ is primary because its radical (x, y) is a maximal ideal.

Primary decomposition

DEFINITION: We say that an ideal $J \subset R$ **admits a primary decomposition** if R is represented as an intersection of primary ideals.

THEOREM: (Noether-Lasker theorem)
Let R be a Noetherian ring. **Then every ideal $J \subset R$ admits a primary decomposition.**

Proof: Indeed, every ideal admits an irreducible decomposition, and irreducible ideals are primary. ■

Emmy Noether

Amalie Emmy Noether (1882-1935).

Emanuel Lasker

Emanuel Lasker (1868-1941).