Commutative Algebra

lecture 5: The ring of G-invariants

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Group representations

DEFINITION: Representation of a group G is a homomorphism $G \longrightarrow GL(V)$. In this case, V is called **representation space**, and **a representation**. We consider V as a vector space with the linear action of a group G. **A morphism of G-representations** is a linear map compatible with the G-action.

DEFINITION: Irreducible representation of G is a representation having no G-invariant subspaces. **Semisimple representation** is a direct sum of irreducible ones.

Split exact sequences

DEFINITION: An **exact sequence of G-representations** is a sequence of G-representations and moprpusms $... \longrightarrow A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow ...$ such the kernel of each map is the image of the previous one. A **short exact sequence** of G-representations is an exact sequence of form

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0. \quad (*)$$

Here "exact" means that i is injective, j surjective, and image of i is kernel of j. A short exact sequence (*) of G-representations is **split** if there exists a morphism $\varphi: C \longrightarrow B$ of representations such that $\varphi \circ j = \mathrm{Id}_C$. The map φ is called **a section** of the surjective morphism j.

REMARK: Equivalently, **(*) is split when B is decomposed onto a direct sum $B = \operatorname{im} i \oplus C_0$;** in this case j defines an isomorphism $j: C_0 \longrightarrow C$.

EXERCISE: Suppose that any exact sequence of G-representations over a field of characteristic 0 splits. **Prove that any finite-dimensional representation of G is semisimple.**

Semisimplicity of representations of finite groups

PROPOSITION: Let $\operatorname{Rep}_k(G)$ be the category of representations of a finite group G over a field k, with $\operatorname{char}(k)$ coprime with $|G|$. **Then any short exact sequence of G-representations splits.**

Proof. Step 1: Let $0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$ be an exact sequence of G-representations. Choose a basis $\{z_i\}$ in C, and let $\{\tilde z_i\}$ be preimages of z_i in B. Axiom of Choice gives a way to chose these preimages even if the set $\{z_i\}$ is infinite. Let $\varphi: C \longrightarrow B$ take z_i to $\tilde z_i$. Then $B = i(A) \oplus \varphi(C)$. However, **this does not imply that (*) splits, because the map φ is not necessarily G-invariant, and the space $\varphi(C)$ is not necessarily a subrepresentation.**

Step 2: We are going to modify φ such that it becomes G-invariant. Consider the action of G on $\operatorname{Hom}(C,B)$ taking $g \in G$ and $u \in \operatorname{Hom}(C,B)$ to $gug^{-1} \in \operatorname{Hom}(C,B)$; here the first "$g$" denotes the corresponding element in $GL(B)$ and the "g^{-1}" denotes the element in $GL(C)$. **Then φ is a morphism of G-representations if and only if φ is G-invariant.**

Semisimplicity of representations of finite groups (2)

PROPOSITION: Let $\operatorname{Rep}_k(G)$ be the category of representations of a finite group G over a field k, with $\operatorname{char}(k)$ coprime with $|G|$. **Then any short exact sequence of G-representations splits.**

Proof. Step 1: Let $0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$ be an exact sequence of G-representations. Consider j as a surjection of vector spaces and **find a section $\varphi : C \longrightarrow B$ (not necessarily G-invariant) using a basis in C.**

Step 2: To split this exact sequence of representations, **φ should be chosen G-invariant.**

Step 3: Since $\operatorname{char} k$ is coprime with $|G|$, the number $|G|$ is invertible in k. Let $\varphi_0 := \frac{1}{|G|}\sum_{g\in G} g(\varphi)$. **This is a sum of all elements in a G-orbit,** hence it is G-invariant. For any $v \in C$, one has

$$i(\varphi_0(v)) = \frac{1}{|G|}\sum_{g\in G} j(g(\varphi))(g^{-1}v) = \frac{1}{|G|}\sum_{g\in G} g(j\varphi((g^{-1}v))) = \frac{1}{|G|}\sum_{g\in G} g(g^{-1}(v)) = v,$$

because j commutes with φ. This implies that **φ_0 is a G-invariant section of j.** ■

COROLLARY: Let G be a finite group and k a field, $\operatorname{char} k$ coprime with $|G|$. **Then any finite-dimensional representation of G over k is semisimple.**

Exact functors (reminder)

DEFINITION: A functor $A \longrightarrow FA$ on the category of R-modules or vector spaces is called **left exact** if any exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is mapped to an exact sequence

$$0 \longrightarrow FA \longrightarrow FB \longrightarrow FC,$$

right exact if it is mapped to an exact sequence

$$FA \longrightarrow FB \longrightarrow FC \longrightarrow 0,$$

and **exact** if the sequence

$$0 \longrightarrow FA \longrightarrow FB \longrightarrow FC \longrightarrow 0$$

is exact.

DEFINITION: Let G be a finite group, and V its representation. Define **the space of G-invariants** V^G as the space of all G-invariant vectors, and **the space of coinvariants** as the quotient of V by its subspace generated by vectors $v - g(v)$, where $g \in G, v \in V$.

EXERCISE: Prove that **the functor $V \longrightarrow V^G$ is left exact, and $V \longrightarrow V_G$ is right exact**.

The functor $V \longrightarrow V^G$ is exact

CLAIM: Let V be an irreducible representation of G. **Then its invariants and co-ivariants are equal 0 if it is non-trivial, and equal V if it is trivial.**

COROLLARY: Let V be a semisimple representation of G. **Then $V_G = V^G$.**

COROLLARY: For any finite group G, the functor of G-invariants is exact.

REMARK: The averaging map

$$m \longrightarrow \frac{1}{|G|} \sum_{g \in G} g(m)$$

gives a projection of V to V^G, and the kernel of this map is the kernel of the natural projection $V \longrightarrow V_G$

Noether theorem (scheme of the proof)

THEOREM: Let R be a finitely generated ring over $\mathbb{C}$, and G a finite group acting on R by automorphisms. Then **the ring R^G of G-invariants is finitely generated**.

Scheme of the proof:

1. Noetheriannes of R is used to prove that R^G is Noetherian.

2. Prove that R^G is finite generated for $R = \mathbb{C}[z_1, ..., z_n]$, where R acts on polynomials of degree 1 by linear automorphisms.

3. Deduce the general case from (2) and exactness of $V \longrightarrow V^G$

Emmy Noether (1882-1935)

Emmy Noether,
illustration by María Castelló Solbes

Ideals in R and R^G

LEMMA: Let R be a ring, G a finite group acting on R, R^G the ring of G-invariants, and $I \subset R^G$ an ideal. Then **the ideal RI satisfies** $\operatorname{Av}_G(RI) = \operatorname{Av}_G(R)I = R^G I = I$**,** where $\operatorname{Av}_G : R \longrightarrow R^G$ denotes the averaging map. ■

COROLLARY: Let $I_1 \subsetneq I$ be ideals in R^G. **Then $RI_1 \subsetneq RI$.** ■

COROLLARY 1: In these assumptions, **if R is Noetherian, then R^G is also Noetherian.**

Proof: Any infinite, strictly monotonous sequence $I_0 \subsetneq I_1 \subsetneq ...$ of ideals in R^G gives a strictly monotonous sequence $RI_0 \subsetneq RI_1 \subsetneq ...$ in R. ■

Graded rings

DEFINITION: A graded ring is a ring A^*, $A^* = \bigoplus_{i=0}^\infty A^i$, with multiplication which satisfies $A^i \cdot A^j \subset A^{i+j}$ ("grading is multiplicative"). A graded ring is called **of finite type** if all A^i are finitely dimensional.

We will usually assume that A^0 is the base field.

EXAMPLE: Polynomial ring $\mathbb{C}[V] = \bigoplus_i \operatorname{Sym}^i V$ is clearly graded.

Graded rings (2)

Claim 1: Let A^* be a graded ring of finite type. Then **A^* is Noetherian $\Leftrightarrow$ it is finitely generated**.

Proof. Step 1: If A^* is finitely generated, it is Noetherian by Hilbert's basis theorem.

Step 2: Conversely, suppose that A^* is Noetherian. Then the ideal $\bigoplus_{i>0} A^i \subset A^*$ is finitely generated. Let $a_i \in A^{n_i}$ be generators of this ideal over A^*. **We are going to show that products of a_i generate A^*.**

Step 3: Let $z \in A^*$ be a graded element of smallest degree which is not generated by products of a_i. Since a_i generate the ideal $\bigoplus_{i>0} A^i \subset A^*$, we can express z as $z = \sum_i f_i a_i$, where $f_i \in A^*$. However, $\deg f_i < \deg z$, hence all f_i are generated by products of a_i. Then all f_i are generated by products of a_i.
■

A caution: In this argument, two notions of "finitely generated" are present: finitely generated ideals (an additive notion) and finitely generated rings over $\mathbb{C}$ (multiplicative). **These two notions are completely different!** One is defined for ideals (or R-modules), another for a ring over a field. Only the name is the same (bad terminology).

Proof of Noether theorem for polynomial invariants

DEFINITION: Let V be a vector space with basis $z_1, ..., z_n$, and $\mathbb{C}[V] = \bigoplus_i \operatorname{Sym}^i V = \mathbb{C}[z_1, ..., z_n]$ the corresponding polynomial ring. Suppose that G acts on V by linear automorphisms. We extend this action to the symmetric tensors $\bigoplus_i \operatorname{Sym}^i V$ multiplicatively. This implies that G acts on $\mathbb{C}[V]$ by automorphisms. Such action is called **linear**.

CLAIM: (Noether theorem for polynomial invariants)
Let G act linearly on the polynomial ring $\mathbb{C}[V]$. **Then the invariant ring $\mathbb{C}[V]^G$ is finitely generated.**

Proof. Step 1: Since the action of G preserves the grading on $\mathbb{C}[V]$, **the ring $\mathbb{C}[V]^G$ is graded and of finite type.**

Step 2: $\mathbb{C}[V]^G$ is Noetherian, because $\mathbb{C}[V]$ is Noetherian, and the ring of invariants R^G is Noetherian if R is Noetherian (Corollary 1).

Step 3: A finite type Noetherian graded ring is finitely generated by Claim 1. ■

Noether theorem

THEOREM: (Noether theorem)
Let R be a finitely generated ring over $\mathbb{C}$, and G a finite group acting on R by automorphisms. Then **the ring R^G of G-invariants is finitely generated**.

Proof. Step 1: Let $f_1, ..., f_m$ be generators of R, and $\{g_1, ..., g_k\} = G$. Consider the space $V \subset R$ generated by all vectors $g_i f_j$. Clearly, $V \subset R$ is V-invariant, and **the natural homomorphism $\mathbb{C}[V] \longrightarrow R = \mathbb{C}[V]/I$ is surjective and G-invariant.**

Step 2: The natural map $\mathbb{C}[V]^G \longrightarrow R^G$ is surjective, because the functor $W \longrightarrow W^G$ is exact.

Step 3: The ring $\mathbb{C}[V]^G$ is finitely generated by Noether theorem for polynomial invariants, hence its quotient R^G is also finitely generated. ■