Commutative Algebra

lecture 6: Tensor product

Misha Verbitsky

http://verbit.ru/IMPA/CA-2026/

IMPA, sala 236

April 6, 2026

Tensor product

DEFINITION: Let R be a ring, and M, M' modules over R. We denote by $M \otimes_R M'$ an R-module generated by symbols $m \otimes m'$, $m \in M, m' \in M'$, modulo relations

$$\begin{gathered} r(m \otimes m') = (rm) \otimes m' = m \otimes (rm'), \\ (m + m_1) \otimes m' = m \otimes m' + m_1 \otimes m', \\ m \otimes (m' + m'_1) = m \otimes m' + m \otimes m'_1 \end{gathered}$$

for all $r \in R, m, m_1 \in M, m', m'_1 \in M'$. Such an R-module is called **the tensor product of M and M' over R.**

REMARK: Suppose that M is generated over R by a set $\{m_i \in M\}$, and M' generated by $\{m'_j \in M'\}$. **Then $M \otimes_R M'$ is generated by the set $\{m_i \otimes m'_j\}$.**

EXERCISE: Find **two non-zero R-modules A, B such that $A \otimes_R B = 0$** when

a. $R = \mathbb{Z}$.

b. $R = C^\infty M$ the ring of smooth functions on a manifold.

c. $R = \mathbb{C}[t]$ (polynomial ring).

Hermann Grassmann (1809 - 1877)

Hermann Grassmann, *Die lineale Ausdehnungslehre, ein neuer Zweig der Mathematik* (Linear extension theory, a new branch of mathematics), 1844.

...Thirty years after the publication of A1 the publisher wrote to Grassmann: "Your book Die Ausdehnungslehre has been out of print for some time. Since your work hardly sold at all, roughly 600 copies were used in 1864 as waste paper and the remaining few odd copies have now been sold out, with the exception of the one copy in our library"... (Prasolov, Viktor V., Problems and Theorems in Linear Algebra.)

Hassler Whitney (1907 - 1989)

Hassler Whitney, 1924

Whitney, Hassler. Tensor products of Abelian groups. Duke Mathematical Journal 4 (1938), no. 3, 495–528.

Hassler Whitney (1907 - 1989)

There was a further block to my progress. I had to handle tensors; but how could I when I was not permitted to *see* them, being only allowed to *learn* about their changing costumes under changes of coordinates? I had somehow to grab the rascals, and look straight at them. I could *look at* a pair of vectors, "multiplied": $u \vee v$. And here, I needed $u \vee v = -v \vee u$. So I managed to construct the rest of the beasts, in "tensor products of abelian groups." (*Duke Math Journal*, 1938). Before long I noticed that neat form, using less space, was the sine qua non of mathematical writing: the CORRECT definition of the tensor product of two vector spaces must use the linear functionals over the linear functionals over one of them. So this is the way in which later generations learned them.

Whitney's "Collected papers", vol. 1

Nicolas Bourbaki

DEFINITION 1. *The tensor product of the right* A*-module* E *and the left* A*-module* F, *denoted by* $E \underset{A}{\otimes} F$ *or* $E \otimes_A F$ (or simply $E \otimes F$ if no confusion is to be feared) *is the quotient* **Z***-module* C/D (the quotient of the **Z**-module C of formal linear

243

II LINEAR ALGEBRA

combinations of elements of $E \times F$ with coefficients in **Z**, by the submodule D generated by the elements of one of the types (2)). *For* $x \in E$ *and* $y \in F$, *the element of* $E \otimes_A F$ *which is the canonical image of the element* (x, y) *of* $C = \mathbf{Z}^{(E \times F)}$ *is denoted by* $x \otimes y$ *and called the tensor product of* x *and* y.

Nicolas Bourbaki, Algebra, Chapter 2, 1942

Nicolas Bourbaki

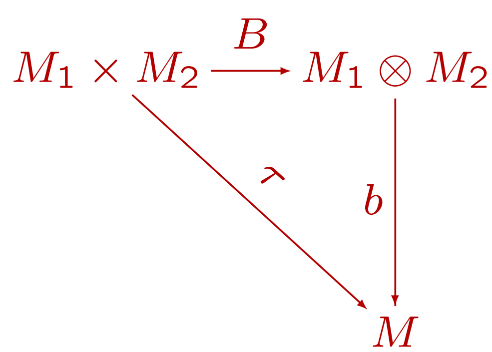
Nicolas Bourbaki

Nicolas Bourbaki, Algebra, Chapter 2, 1942

Bilinear maps

DEFINITION: Let M_1, M_2, M be modules over a ring R. **Bilinear map** $\mu(M_1, M_2) \xrightarrow{\varphi} M$ is a map satisfying $\varphi(rm, m') = \varphi(m, rm') = r\varphi(m, m')$, $\varphi(m + m_1, m') = \varphi(m, m') + \varphi(m_1, m')$, $\varphi(m, m' + m'_1) = \varphi(m, m') + \varphi(m, m'_1)$.

THEOREM: (Universal property of the tensor product)
For any bilinear map $B: M_1 \times M_2 \longrightarrow M$ **there exists a unique homomorphism** $b: M_1 \otimes M_2 \longrightarrow M$**, making the following diagram commutative:**



■

REMARK: If R is the field k, R-modules are vector spaces, and the previous theorem proves that $\operatorname{Bil}(M_1 \times M_2, k) = (M_1 \otimes M_2)^*$. For finite-dimensional M_i, it gives $M_1 \otimes M_2 = (M_1 \otimes M_2)^{**} = \operatorname{Bil}(M_1 \times M_2, k)^*$ (Whitney's definition of tensor product).

Categories (reminder)

DEFINITION: A category $\mathcal{C}$ is a collection of data called "objects" and "morphisms between objects" which satisfies the axioms below.

DATA.

Objects: A class $\mathcal{O}b(\mathcal{C})$ of **objects** of $\mathcal{C}$.

Morphisms: For each $X, Y \in \mathcal{O}b(\mathcal{C})$, one has a set $\mathcal{M}or(X, Y)$ of **morphisms from X to Y**.

Composition of morphisms: For each $\varphi \in \mathcal{M}or(X, Y), \psi \in \mathcal{M}or(Y, Z)$ there exists **the composition** $\varphi \circ \psi \in \mathcal{M}or(X, Z)$

Identity morphism: For each $A \in \mathcal{O}b(\mathcal{C})$ there exists a morphism $\operatorname{Id}_A \in \mathcal{M}or(A, A)$.

AXIOMS.

Associativity of composition: $\varphi_1 \circ (\varphi_2 \circ \varphi_3) = (\varphi_1 \circ \varphi_2) \circ \varphi_3$.

Properties of identity morphism: For each $\varphi \in \mathcal{M}or(X, Y)$, one has $\operatorname{Id}_x \circ \varphi = \varphi = \varphi \circ \operatorname{Id}_Y$

Categories 2 (reminder)

DEFINITION: Let $X, Y \in \mathcal{O}b(\mathcal{C})$ be objects of $\mathcal{C}$. A morphism $\varphi \in \mathcal{M}or(X,Y)$ is called **an isomorphism** if there exists $\psi \in \mathcal{M}or(Y,X)$ such that $\varphi \circ \psi = \operatorname{Id}_X$ and $\psi \circ \varphi = \operatorname{Id}_Y$. In this case, the objects X and Y are called **isomorphic**.

Examples of categories:

Category of sets: its morphisms are arbitrary maps.

Category of vector spaces: its morphisms are linear maps.

Categories of rings, groups, fields: morphisms are homomorphisms.

Category of topological spaces: morphisms are continuous maps.

Category of smooth manifolds: morphisms are smooth maps.

Le Bourgeois gentilhomme

Monsieur Jourdain. Et comme l'on parle qu'est-ce que c'est donc que cela?

Maître de philosophie. De la prose.

Monsieur Jourdain. Quoi? quand je dis: "Nicole, apportez-moi mes pantoufles, et me donnez mon bonnet de nuit", c'est de la prose?

Maître de philosophie. Oui, Monsieur.

Monsieur Jourdain. Par ma foi! il y a plus de quarante ans que je dis de la prose sans que j'en susse rien, et je vous suis le plus obligé du monde de m'avoir appris cela.

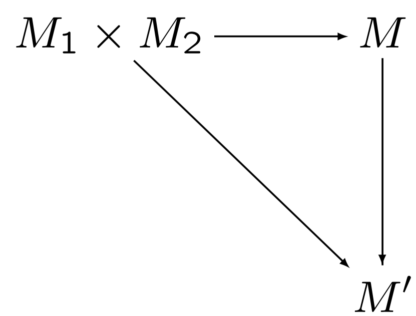
Universal property of the tensor product and categories

DEFINITION: Initial object of a category $\mathcal{C}$ is an object $X \in \mathcal{Ob}(\mathcal{C})$ such that for any $Y \in \mathcal{Ob}(\mathcal{C})$ there exists a unique morphism $X \longrightarrow Y$.

EXAMPLE: Zero space **is an initial object in the category of vector spaces.** The ring $\mathbb{Z}$ **is an initial object in the category of rings with unit.**

EXERCISE: Prove that **initial object is unique.**

DEFINITION: Let M_1, M_2 are R-modules, and $\mathcal{C}$ the following category. Objects of $\mathcal{C}$ are pairs (R-module M, bilinear map $M_1 \times M_2 \longrightarrow M$). Morphisms of $\mathcal{C}$ are homomorphisms $M \xrightarrow{\varphi} M'$ making the following diagram commutative:



CLAIM: (Universal property of the tensor product)
Tensor product $M_1 \times M_2$ is the initial object in $\mathcal{C}$.

COROLLARY: Tensor product is uniquely determined by the universal property.

Indeed, the initial object is unique.

The internal $\mathcal{H}om$ and exact functors

DEFINITION: Let M, M' be R-modules. Consider the group $\operatorname{Hom}_R(M, M')$ of R-module homomorphisms. We consider $\operatorname{Hom}_R(M, M')$ as an R-module, using $r\varphi(m) := \varphi(rm)$. This R-module is called **the internal** Hom **functor**, denoted $\mathcal{H}om$, or $\mathcal{H}om_R$.

Claim 1: Let $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$ be an exact sequence of R-modules. **Then the natural sequences**

$$0 \longrightarrow \mathcal{H}om_R(M_3, N) \longrightarrow \mathcal{H}om_R(M_2, N) \longrightarrow \mathcal{H}om_R(M_1, N)$$

and

$$0 \longrightarrow \mathcal{H}om_R(N, M_1) \longrightarrow \mathcal{H}om_R(N, M_2) \longrightarrow \mathcal{H}om_R(N, M_3)$$

are exact, for any R-module M.

Proof: Let's prove exactness of the first sequence. Exactness in the term $\mathcal{H}om_R(M_3, N)$ is clear. If $\nu \in \mathcal{H}om_R(M_2, N)$ is mapped to 0 in projection to $\mathcal{H}om_R(M_1, N)$, this means that $\nu|_{M_1} = 0$, giving a morphism $\tilde{\nu} \in \mathcal{H}om_R(M_3, N)$, which is mapped to ν. **Exactness of the second sequence is left as an exercise.** ■

The internal $\mathcal{H}om$ and tensor product

REMARK: Universal property of $\otimes_R$ implies

$$\mathcal{H}om_R(M_1 \otimes_R M_2, M) = \mathcal{H}om_R(M_1, \mathcal{H}om_R(M_2, M)).$$

Indeed, **the group $\mathcal{H}om_R(M_1, \mathcal{H}om_R(M_2, M))$ is identified with the group of bilinear maps $M_1 \times M_2 \longrightarrow M$.**

COROLLARY: Let $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$ be an exact sequence of R-modules. **Then for any R-modules N, N', the sequence**

$$0 \longrightarrow \mathcal{H}om_R(M_3 \otimes N', N) \longrightarrow \mathcal{H}om_R(M_2 \otimes N', N) \longrightarrow \mathcal{H}om_R(M_1 \otimes N', N)$$

is exact.

Proof: Using Claim 1 twice, we obtain an exact sequence

$$\begin{aligned} 0 \longrightarrow \mathcal{H}om_R(N', \mathcal{H}om_R(M_3, N)) \\ \longrightarrow \mathcal{H}om_R(N', \mathcal{H}om_R(M_2, N)) \longrightarrow \mathcal{H}om_R(N', \mathcal{H}om_R(M_3, N)). \end{aligned}$$

Then we use an isomorphism $\mathcal{H}om_R(A \otimes_R B, M) = \mathcal{H}om_R(A, \mathcal{H}om_R(B, M))$ proven above. ■

Functor $\mathcal{H}om$, part 2

REMARK: Exactness of the sequence $M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$ implies exactness of $0 \longrightarrow \mathcal{H}om_R(M_3, N) \longrightarrow \mathcal{H}om_R(M_2, N) \longrightarrow \mathcal{H}om_R(M_1, N)$. We are going to prove the converse: **exactness of the second sequence (for all N) implies exactness of the first one.**

DEFINITION: A complex of R-modules is a sequence $M_1 \xrightarrow{d_1} M_2 \xrightarrow{d_2} M_3 \xrightarrow{d_3} ...$ such that $d_i \circ d_{i+1} = 0$.

LEMMA: Consider a complex E^* of R-modules $M_1 \xrightarrow{\mu} M_2 \xrightarrow{\rho} M_3 \longrightarrow 0$ such that $0 \longrightarrow \mathcal{H}om_R(M_3, N) \xrightarrow{\rho_N} \mathcal{H}om_R(M_2, N) \xrightarrow{\mu_N} \mathcal{H}om_R(M_1, N)$ is exact for all N. **Then E^* is also exact.**

Proof: Injectivity of ρ_N implies surjectivity of ρ, if we put $N = M_3/\operatorname{im}\rho$. Exactness of the second sequence in term $\mathcal{H}om_R(M_2, N)$ implies exactness of E in term M_2 when $N = M_2/\operatorname{im}\mu$. ■

Exactness of the tensor product

THEOREM: Let $M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$ be an exact sequence of R-modules. **Then the sequence**

$$M_1 \otimes_R M \longrightarrow M_2 \otimes_R M \longrightarrow M_3 \otimes_R M \longrightarrow 0 \qquad (*)$$

is exact.

Proof: Using the universal property of tensor product, we have shown that

$$0 \longrightarrow \mathcal{H}om_R(M_3 \otimes M, N) \longrightarrow \mathcal{H}om_R(M_2 \otimes M, N) \longrightarrow \mathcal{H}om_R(M_1 \otimes M, N)$$

is exact for any N. Aplying the previous lemma, we obtain that (*) is also exact. ■

COROLLARY: Let $I \subset R$ be an ideal in a ring. **Then $M \otimes_R (R/I) = M/IM$.**

Proof: Apply the functor $\otimes_R M$ to the exact sequence $0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$. We obtain $IM \longrightarrow M \longrightarrow (R/I) \otimes_R M \longrightarrow 0$. ■

Tensor product: examples

EXERCISE: Prove that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} = 0$.

REMARK: Let $\mathbb{Z} \xrightarrow{\varphi} Z$ be a multiplication by 2. Then the sequence

$$\mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\varphi} \mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z}) \longrightarrow (\mathbb{Z}/2\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z}) \longrightarrow 0$$

obtained from $0 \longrightarrow \mathbb{Z} \xrightarrow{\varphi} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$ by tensoring with $\otimes_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z})$ **is not left exact**.