Commutative Algebra

lecture 7: Tensor products of rings and products of varieties

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Equivalence of functors (reminder)

DEFINITION: Let $X, Y \in \mathcal{O}b(\mathcal{C})$ be objects of a category $\mathcal{C}$. A morphism $\varphi \in \mathcal{M}or(X,Y)$ is called **an isomorphism** if there exists $\psi \in \mathcal{M}or(Y,X)$ such that $\varphi \circ \psi = \mathrm{Id}_X$ and $\psi \circ \varphi = \mathrm{Id}_Y$. In this case X and Y are called **isomorphic**.

DEFINITION: Two functors $F, G : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ are called **equivalent** if for any $X \in \mathcal{O}b(\mathcal{C}_1)$ we are given an isomorphism $\Psi_X : F(X) \longrightarrow G(X)$, in such a way that for any $\varphi \in \mathcal{M}or(X,Y)$, one has $F(\varphi) \circ \Psi_Y = \Psi_X \circ G(\varphi)$.

REMARK: Such commutation relations are usually expressed by **commutative diagrams**. For example, the condition $F(\varphi) \circ \Psi_Y = \Psi_X \circ G(\varphi)$ is expressed by a commutative diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{F(\varphi)} & F(Y) \\ {\scriptstyle \Psi_X}\downarrow & & \downarrow{\scriptstyle \Psi_Y} \\ G(X) & \xrightarrow{G(\varphi)} & G(Y) \end{array}$$

Equivalence of categories (reminder)

DEFINITION: A functor $F: \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ is called **equivalence of categories** if there exists a functor $G: \mathcal{C}_2 \longrightarrow \mathcal{C}_1$ such that the compositions $G \circ F$ and $G \circ F$ are equivaleent to the identity functors $\operatorname{Id}_{\mathcal{C}_1}$, $\operatorname{Id}_{\mathcal{C}_2}$.

REMARK: It is possible to show that this is equivalent to the following conditions: F **defines a bijection on the set of isomorphism classes of objects of** $\mathcal{C}_1$ **and** $\mathcal{C}_2$**, and a bijection**

$$\mathcal{M}or(X,Y) \longrightarrow \mathcal{M}or(F(X),F(Y)).$$

for each $X, Y \in \mathcal{O}b(\mathcal{C}_1)$.

REMARK: From the point of view of category theory, **equivalent categories are two instances of the same category** (even if the cardinality of corresponding sets of objects is different).

Categories and Hilbert Nullstellensatz (reminder)

DEFINITION: Category of affine varieties over $\mathbb{C}$: its objects are algebraic subsets in $\mathbb{C}^n$, morphisms – polynomial maps.

DEFINITION: Finitely generated ring over $\mathbb{C}$ is a quotient of $\mathbb{C}[t_1, ..., t_n]$ by an ideal.

DEFINITION: Let R be a ring. An element $x \in R$ is called **nilpotent** if $x^n = 0$ for some $n \in \mathbb{Z}^{>0}$.

Theorem: Let $\mathcal{C}_R$ be a category of finitely generated rings over $\mathbb{C}$ without non-zero nilpotents and $\mathcal{A}\!f\!f$ – category of affine varieties. Consider the functor $\Phi: \mathcal{A}\!f\!f \longrightarrow \mathcal{C}_R^{op}$ mapping an algebraic variety X to the ring of polynomial functions on X. **Then Φ is an equivalence of categories.**

Proof: Follows from Hilbert Nullstellensatz (Lecture 2). ■

Tensor product of rings

DEFINITION: Let A, B be rings, $C \longrightarrow A$, $C \longrightarrow B$ homomorphisms. Consider A and B as C-modules, and let $A \otimes_{\mathbb{C}} B$ be their tensor product. Define the multiplication on $A \otimes_C B$ as $a \otimes b \cdot a' \otimes b' = aa' \otimes bb'$. By universal property, this map is extended to a product on $A \otimes_C B$. This defines **the tensor product of rings**.

EXAMPLE: $\mathbb{C}[t_1, ..., t_k] \otimes_{\mathbb{C}} \mathbb{C}[z_1, ..., z_n] = \mathbb{C}[t_1, ..., t_k, z_1, ..., z_n]$. Indeed, if we denote by $\mathbb{C}_d[t_1, ..., t_k]$ the space of homogeneous polynomials of degree d, then $\mathbb{C}_d[t_1, ..., t_k] \otimes_{\mathbb{C}} \mathbb{C}_{d'}[z_1, ..., z_n]$ is polynomials of degree d in $\{t_i\}$ and d' in $\{z_i\}$.

EXAMPLE: For any homomorphism $\varphi : C \longrightarrow A$, **the ring $A \otimes_C (C/I)$ is a quotient of A by the ideal $A \cdot \varphi(I)$**. This follows from $M \otimes_R (R/I) = M/IM$.

PROPOSITION: (associativity of $\otimes$)
Let $C \longrightarrow A, C \longrightarrow B, C' \longrightarrow B, C' \longrightarrow D$ be ring homomorphisms. **Then $(A \otimes_C B) \otimes_{C'} D = A \otimes_C (B \otimes_{C'} D)$.**
Proof: Universal property of $\otimes$ implies that $\operatorname{Hom}((A \otimes_C B) \otimes_{C'} D, M) = \operatorname{Hom}(A \otimes_C (B \otimes_{C'} D), M)$ is the space of trilinear maps $A, B, D \longrightarrow M$ satisfying $\varphi(ca, b, d) = \varphi(a, cb, d)$ and $\varphi(a, c'b, d) = \varphi(a, b, c'd)$. However, an object X of category is defined by the functor $\cdot \mapsto \mathfrak{Mor}(X, \cdot)$ uniquely up to isomorphism **(prove it)**. ■

Tensor product of rings and preimage of a point

DEFINITION: Recall that **the spectrum** of a finitely generated ring R is the corresponding algebraic variety, denoted by $\operatorname{Spec}(R)$

PROPOSITION: Let $f: X \longrightarrow Y$ be a morphism of affine varieties, $f^*: \mathcal{O}_Y \longrightarrow \mathcal{O}_X$ the corresponding ring homomorphism, $y \in Y$ a point, and $\mathfrak{m}_y$ its maximal ideal. **Denote by R_1 the quotient of $R := \mathcal{O}_X \otimes_{\mathcal{O}_Y} (\mathcal{O}_Y/\mathfrak{m}_y)$ by its nilradical. Then $\operatorname{Spec}(R_1) = f^{-1}(y)$.**

Proof. Step 1: If $\alpha \in \mathcal{O}_Y$ vanishes in y, $f^*(\alpha)$ vanishes in all points of $f^{-1}(y)$. This implies that **the set V_I of common zeros of the ideal $I := \mathcal{O}_X \cdot f^*\mathfrak{m}_y$ contains $f^{-1}(y)$.**

Step 2: If $f(x) \neq y$, take a function $\beta \in \mathcal{O}_Y$ vanishing in y and non-zero in $f(x)$. Since $\varphi^*(\beta)(x) \neq 0$ and $\beta(y) = 0$, this gives $x \notin V_I$. **We proved that the set of common zeros of the ideal $I = \mathcal{O}_X \cdot f^*\mathfrak{m}_y$ is equal to $f^{-1}(y)$.**

Step 3: Now, strong Nullstellensatz implies that $\mathcal{O}_{f^{-1}(y)}$ is a quotient of $R = \mathcal{O}_X/I$ by nilradical. ■

Tensor product of rings and product of varieties

LEMMA: For any rings A, B, C, there is a natural isomorphism $A \otimes_C B \otimes_B B' = A \otimes_C B'$.
Proof: Follows from associativity of tensor product and $B \otimes_B B' = B'$. ■

LEMMA: $A \otimes_C (B/I) = A \otimes_C B/(1 \otimes I)$, where $1 \otimes I$ denotes the ideal $A \otimes_C I$.
Proof: Using $M \otimes_R (R/I) = M/IM$ (Lecture 6), we obtain

$$A \otimes_C (B/I) = (A \otimes_C B) \otimes_B (B/I) = (A \otimes_C B)/(1 \otimes I) \quad ■$$

Lemma 1: Let A, B be finitely generated rings without nilpotents, $R := A \otimes_{\mathbb{C}} B$, and $N \subset R$ nilradical. **Then** $\operatorname{Spec}(R/N) = \operatorname{Spec}(A) \times \operatorname{Spec}(B)$.

Proof. Step 1: Let $A = \mathbb{C}[t_1, ..., t_n]/I, B = \mathbb{C}[z_1, ..., z_k]/J$. Then $\mathbb{C}[t_1, ..., t_n] \otimes_{\mathbb{C}} \mathbb{C}[z_1, ..., z_k] = \mathbb{C}[t_1, ..., t_n, z_1, ..., z_k]$. Applying the previous lemma twice, **we obtain** $A \otimes_{\mathbb{C}} B = \mathbb{C}[t_1, ..., t_n, z_1, ..., z_k]/(I + J)$. Here $I + J$ means $I \otimes 1 \oplus 1 \otimes J$.

Step 2: The set V_{I+J} of common zeros of $I + J$ is $\operatorname{Spec}(A) \times \operatorname{Spec}(B) \subset \mathbb{C}^n \times \mathbb{C}^k$.

Step 3: Hilbert Nullstellensatz implies $\operatorname{Spec}(R/N) = V_{I+J} = \operatorname{Spec}(A) \times \operatorname{Spec}(B)$. ■

Tensor product of rings and product of varieties (2)

LEMMA: For any finitely-generated ring A over $\mathbb{C}$, **intersection P of all its maximal ideals is its nilradical.**

Proof: Let $A = \mathbb{C}[t_1, ..., t_n]/I$, and $Z = V_I$ the set of common zeros. Strong Nullstellensatz implies that $f \in A$ **is nilpotent if and only if $f = 0$ in each point of Z.** This is equivalent to $f \in P$. ■

REMARK: Let A, B be finite generated rings over $\mathbb{C}$, $B \longrightarrow A$ a homomorphism, and $\mathfrak{m} \subset B$ a maximal ideal. Then the ring $A \otimes_B (B/\mathfrak{m})$ **can contain nilpotents**, even if A and B have no zero divisors.

EXERCISE: Give an example of such rings A, B.

THEOREM: Let A, B be finitely-generated, reduced rings over $\mathbb{C}$, and $R := A \otimes_{\mathbb{C}} B$ their product. **Then R is reduced** (that is, has no nilpotents).

Proof: see the next slide.

COROLLARY: Spec$(A) \times$ Spec$(B) =$ Spec$(A \otimes_{\mathbb{C}} B)$.

Tensor product of rings and product of varieties (3)

THEOREM: Let A, B be finitely-generated, reduced rings over $\mathbb{C}$, and $R := A \otimes_{\mathbb{C}} B$ their product. **Then R is reduced**.

Proof. Step 1: By the previous lemma, it suffices to show that the **intersection P of maximal ideals of R is 0.**

Step 2: Let X, Y denote the varieties $\operatorname{Spec}(A), \operatorname{Spec}(B)$. Lemma 1 implies that **maximal ideals of R are points of $X \times Y$.**

Step 3: Every such ideal is given as $\mathfrak{m}_x \otimes \mathcal{O}_Y + \mathcal{O}_X \otimes \mathfrak{m}_y$, where $x \in X, y \in Y$. Then

$$P = \bigcap_{(x,y)\in X\times Y} (\mathfrak{m}_x \otimes \mathcal{O}_Y + \mathcal{O}_X \otimes \mathfrak{m}_y) = \bigcap_{y\in Y} \left(\left(\bigcap_{x\in X} \mathfrak{m}_x \otimes \mathcal{O}_Y \right) + \mathcal{O}_X \otimes \mathfrak{m}_y \right) = \bigcap_Y \mathcal{O}_X \otimes \mathfrak{m}_y = 0.$$

This follows from $\bigcap_Y 1 \otimes \mathfrak{m}_y = \bigcap_X \mathfrak{m}_x \otimes 1 = 0$ since A and B are reduced. ∎

Preimage and diagonal

Claim 2: Let $f: X \longrightarrow Y$ be a morphism of algebraic varieties, $f^*: \mathcal{O}_Y \longrightarrow \mathcal{O}_X$ the corresponding ring homomorphism, $Z \subset Y$ a subvariety, and I_Z its ideal. Denote by R_1 the quotient of a ring $R := \mathcal{O}_X \otimes_{\mathcal{O}Y} (\mathcal{O}_Y/I_Z) = \mathcal{O}_X/f^*(I_Z)$ by its nilradical. **Then** $\operatorname{Spec}(R_1) = f^{-1}(Z)$.

Proof: Clearly, the set of common zeros of the ideal $J := f^*(I_Z)$ contains $f^{-1}(Z)$. On the other hand, for any point $x \in X$ such that $f(x) \notin Z$ there exist a function $g \in J$ such that $g(x) \neq 0$. Therefore, $f^{-1}(Z) = V_J$, and strong Nullstellensatz implies that $\mathcal{O}_{f^{-1}(Z)} = R_1$. ■

Claim 3: Let M be an algebraic variety, $\Delta \subset M \times M$ the diagonal, and $I \subset \mathcal{O}_M \otimes_{\mathbb{C}} \mathcal{O}_M$ the ideal generated by $r \otimes 1 - 1 \otimes r$ for all $r \in \mathcal{O}_M$. **Then** $\mathcal{O}_\Delta$ **is** $\mathcal{O}_M \otimes_{\mathbb{C}} \mathcal{O}_M/I$.

Proof. Step 1: By definition of the tensor product, $\mathcal{O}_M \otimes_{\mathbb{C}} \mathcal{O}_M/I = \mathcal{O}_M \otimes_{\mathcal{O}_M} \mathcal{O}_M = \mathcal{O}_M$, hence it is reduced. If we prove that $\Delta = V_I$, the statement of the claim would follow from strong Nullstellensatz.

Step 2: Clearly, $\Delta \subset V_I$. To prove the converse, let $(m, m') \in M \times M$ be a point not on diagonal, and $f \in \mathcal{O}_M$ a function which satisfies $f(m) = 0, f(m') \neq 0$. Then $f \otimes 1 - 1 \otimes f$ is non-zero on (m, m'). ■

Fibered product

DEFINITION: Let $X \xrightarrow{\pi_X} M, Y \xrightarrow{\pi_Y} M$ be maps of sets. **Fibered product** $X \times_M Y$ is the set of all pairs $(x, y) \in X \times Y$ such that $\pi_X(x) = \pi_Y(y)$.

CLAIM: Let $X \xrightarrow{\pi_X} M, Y \xrightarrow{\pi_Y} M$ be morphism of algebraic varieties, $R := \mathcal{O}_X \otimes_{\mathcal{O}_M} \mathcal{O}_Y$, and R_1 the quotient of R by its nilradical. **Then** Spec$(R_1) = X \times_M Y$**.**

Proof: Let I be the ideal of diagonal in $\mathcal{O}_M \otimes_{\mathbb{C}} \mathcal{O}_M$. Since I is generated by $r \otimes 1 - 1 \otimes r$ (Claim 3), $R = \mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{O}_Y / (\pi_X \times \pi_Y)^*(I)$. Applying Claim 2, we obtain that Spec$(R_1) = (\pi_X \times \pi_Y)^{-1}(\Delta)$. ■

Initial and terminal objects

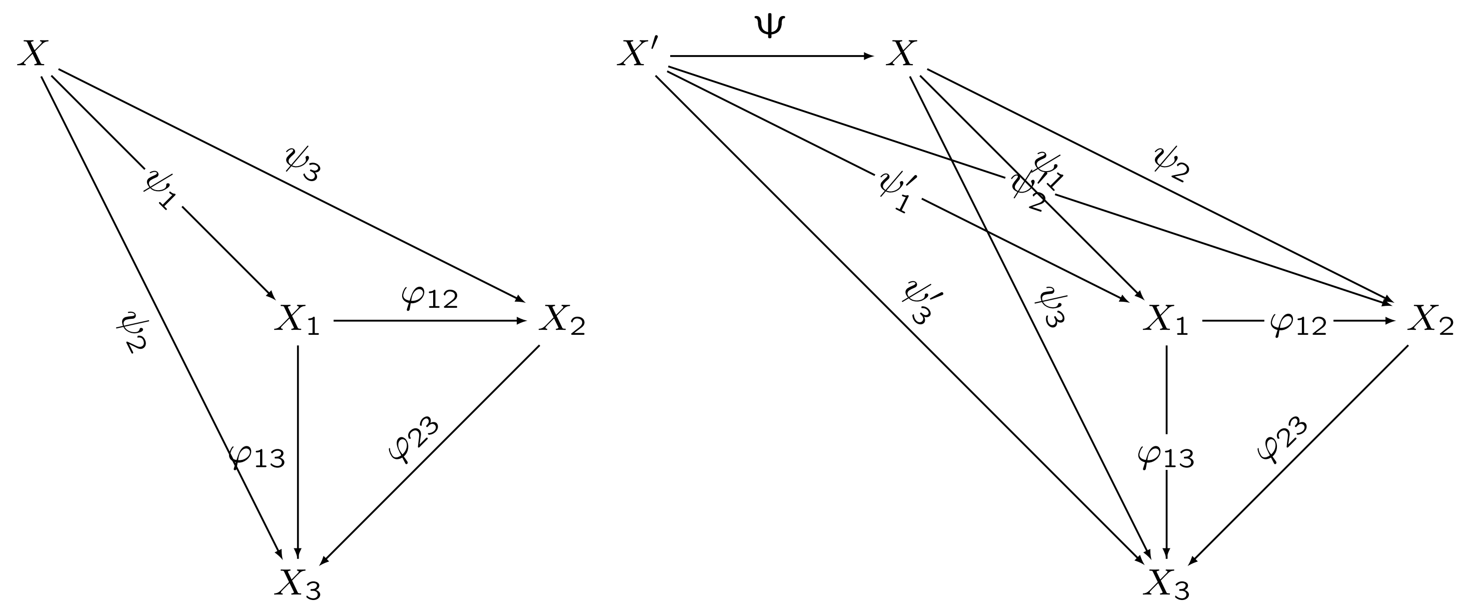
DEFINITION: Commutative diagram in category $\mathcal{C}$ is given by the following data. There is a directed graph (graph with arrows). For each vertex of this graph we have an object of category $\mathcal{C}$, and each arrow corresponds to a morphism of the associated objects. **These morphisms are compatible, in the following way.** Whenever there exist two ways of going from one vertex to another, the compositions of the corresponding arrows are equal.

DEFINITION: An initial object of a category is an object $I \in \mathcal{O}b(\mathcal{C})$ such that $\mathcal{M}or(I, X)$ is always a set of one element. **A terminal object** is $T \in \mathcal{O}b(\mathcal{C})$ such that $\mathcal{M}or(X, T)$ is always a set of one element.

EXERCISE: Prove that **the initial and the terminal object is unique.**

Limits and colimits of diagrams

DEFINITION: Let $S = \{X_i, \varphi_{ij}\}$ be a commutative diagram in $\mathcal{C}$, and $\vec{\mathcal{C}}_S$ be a category of pairs (object X in $\mathcal{C}$, morphisms $\psi_i : X \longrightarrow X_i$, defined for all X_i) making the diagram formed by $(X, X_i, \psi_i, \varphi_{ij})$ commutative.



Morphisms $\mathcal{M}or(\{X, \psi_i\}, \{X', \psi'_i\})$, are morphisms $\Psi \in \mathcal{M}or(X, X')$, making the diagram formed by $(X, X', \psi_i, \psi'_i, \varphi_{ij})$ commutative. The terminal object in this category is called **limit**, or **inverse limit** of the diagram S.

DEFINITION: **Colimit**, or **direct limit** is obtained from the previous definition by inverting all arrows and replacing "terminal" by "initial".

Products and coproducts

EXAMPLE: Let S be a diagram with two vertices X_1 and X_2 and no arrows. The inverse limit of S is called **the product** of X_1 and X_2, and the direct limit **the coproduct**.

EXAMPLE: Products in the category of sets, vector spaces and topological spaces are the usual products of sets, vector spaces and topological spaces **(check this)**.

EXAMPLE: Coproduct in the category of groups is called **free product**, or **amalgamated product**. Coproduct of the group $\mathbb{Z}$ with itself is called **free group**. Coproduct in the category of vector spaces is also the usual product of vector spaces. Coproduct in the category of sets is disjoint union.

Products and coproducts (2)

EXERCISE: Prove that **the product of algebraic varieties is their product in this category.**

EXERCISE: Prove that **coproduct of rings over $\mathbb{C}$ in the category of rings is their tensor product.**

EXERCISE: Prove that **coproduct of reduced rings over $\mathbb{C}$ in the category of reduced rings is the quotient of their tensor product by the nilradical.**

Since the category of algebraic varieties is equivalent to the category of finitely generated reduced rings, this gives another proof of the theorem.

THEOREM: Let A, B be finitely generated reduced rings over $\mathbb{C}$. **Then** $\operatorname{Spec}(A \otimes_{\mathbb{C}} B/I) = \operatorname{Spec}(A) \times \operatorname{Spec}(B)$, where I is the nilradical of $A \otimes_{\mathbb{C}} B$.

Fibered product

DEFINITION: Consider the following diagram:

$$\begin{array}{ccc} A & & B \\ & \searrow \quad \swarrow & \\ & C & \end{array}$$

Its limit is called **fibered product** of A and B over C. Colimit of the diagram

$$\begin{array}{ccc} & C & \\ & \swarrow \quad \searrow & \\ A & & B \end{array}$$

is called **coproduct** of A and B over C.

EXERCISE: Prove that the **fibered product of algebraic varieties is the same as their product in the category of algebraic varieties.**

EXERCISE: Prove that the **coproduct of rings A and B over C is $A \otimes_C B$.** Prove that the **coproduct of reduced rings A and B over C in the category of reduced rings is $A \otimes_C B/I$, where I is nilradical.**

Using strong Nullstellensatz again, we obtain
CLAIM: Let $X \xrightarrow{\pi_X} M, Y \xrightarrow{\pi_Y} M$ be morphisms of affine varieties, $R := \mathcal{O}_X \otimes_{\mathcal{O}_M} \mathcal{O}_Y$, and R_1 the quotient of R by its nilradical. **Then $\operatorname{Spec}(R_1) = X \times_M Y$.**

Natural transformation of functors

DEFINITION: Let $F, G : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ be functors on categories. **A natural transformation of functors** is a morphism $\Psi_X : F(X) \longrightarrow G(X)$ such that for any $\varphi \in \mathcal{M}or(X, Y)$, one has $F(\varphi) \circ \Psi_Y = \Psi_X \circ G(\varphi)$.

REMARK: The condition $F(\varphi) \circ \Psi_Y = \Psi_X \circ G(\varphi)$ **is expressed by a commutative diagram**

$$\begin{array}{ccc} F(X) & \xrightarrow{F(\varphi)} & F(Y) \\ \Psi_X \downarrow & & \downarrow \Psi_Y \\ G(X) & \xrightarrow{G(\varphi)} & G(Y) \end{array}$$

REMARK: Equivalence of functors **is a special case of a natural transformation of functors.**

Representable functors and natural transformations

DEFINITION: Consider the functor $h_A: \mathcal{C} \longrightarrow \mathcal{S}ets$ taking $X \in \mathcal{O}b(\mathcal{C})$ to $\mathcal{M}or(A, X)$. We say that h_A **is represented by an object** $A \in \mathcal{O}b(\mathcal{C})$.

CLAIM: Let $\Phi: h_A \longrightarrow F$ be a natural transformation of functors from $\mathcal{C}$ to sets. **Then Φ is uniquely determined by the element $\Phi(\mathrm{Id}_A) \in F(A)$.**

Proof: For any $\lambda \in \mathcal{M}or(A, B)$, we have a commutative diagram

$$\begin{array}{ccc} h_A(A) = \mathcal{M}or(A,A) & \xrightarrow{f \mapsto f\circ\lambda} & h_A(B) = \mathcal{M}or(A,B) \\ \Phi_A \downarrow & & \downarrow \Phi_B \\ F(A) & \xrightarrow{F(\lambda)} & F(B) \end{array}$$

Suppose that the top arrow takes Id_A to λ. Commutativity of this diagram implies that $\Phi_B(\lambda) = F(\lambda)(\Phi_A(\mathrm{Id}_A))$, hence **the map $\Phi_B: \mathcal{M}or(A, B) \longrightarrow F(B)$ is uniquely determined by $\Phi_A(\mathrm{Id}_A) \in \mathcal{M}or(A, A)$.** ■

Yoneda lemma

This brings the following useful result.

THEOREM: (Yoneda lemma)
Let $\mathcal{C}$ be a category, $A \in \mathcal{O}b(\mathcal{C})$, and $h_A : \mathcal{C} \longrightarrow \mathcal{S}ets$ the functor represented by A. Consider a functor $F : \mathcal{C} \longrightarrow \mathcal{S}ets$. Then **the set of natural transformations $h_A \longrightarrow F$ is in bijective correspondence with $F(A)$.** ■

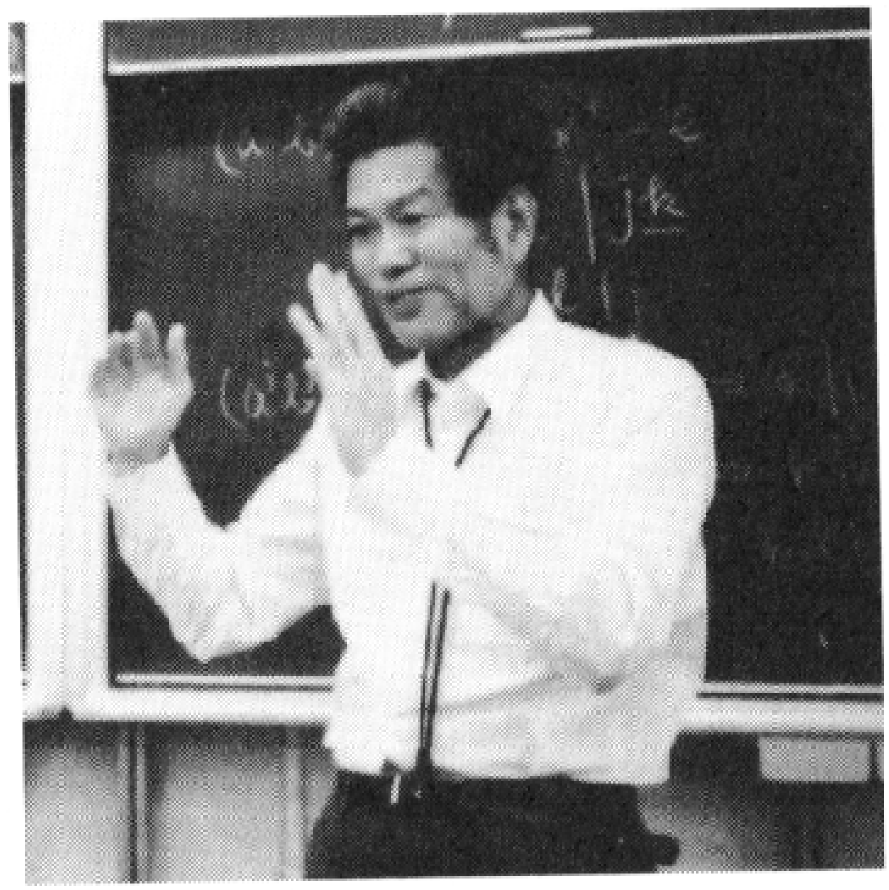
The functors $F : \mathcal{C} \longrightarrow \mathcal{S}ets$ form a category. Objects of this category are functors $F : \mathcal{C} \longrightarrow \mathcal{S}ets$, morphisms are natural transforms. Yoneda lemma **immediately implies that $\mathcal{M}or(h_A, h_B) = \mathcal{M}or(B, A)$.** We obtained the following statement.

CLAIM: Let $\mathcal{C}$ be a category, and $\mathcal{F}$ the category of representable functors $F : \mathcal{C} \longrightarrow \mathcal{S}ets$. **Then the contravariant functor $A \longrightarrow h_A$ defines an equivalence of categories $\mathcal{C} \longrightarrow \mathcal{F}^\circ$.** ■

REMARK: In particular, **an object A of category $\mathcal{C}$ representing a given functor $h_A : \mathcal{C} \longrightarrow \mathcal{S}ets$ is uniquely up to an isomorphism determined by this functor**.

Yoneda lemma (2)

REMARK: The same is true for contravariant functors: **a category $\mathcal{C}$ is equivalent to the category $\mathcal{G}$ of contravariant functors $\mathcal{C}^\circ \longrightarrow \mathcal{S}ets$ represented by $h_A^\circ(X) = \mathcal{M}or(X, A)$.**



米田信夫

Yoneda Nobuo, (1930-1996)

Products and coproducts (3)

EXAMPLE: By definition, **the product $A \times B$ in a category is the object representing the functor $X \to \mathfrak{Mor}(X, A) \times \mathfrak{Mor}(X, B)$.** Similarly, **the product $A \coprod B$ in a category is the object representing the functor $X \to \mathfrak{Mor}(A, X) \times \mathfrak{Mor}(B, X)$.**