

Commutative Algebra

lecture 8: Artinian rings

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Field extensions

DEFINITION: An extension of a field k is a field K containing k . We write “ K is an extension of k ” as $[K : k]$.

DEFINITION: Let $k \subset K$ be a field contained in a field. In this case, we say that k is a subfield of K , and K is an extension of k . An element $x \in K$ is called algebraic over K if x is a root of a non-zero polynomial with coefficients in k . An element which is not algebraic is called transcendental.

THEOREM: A sum and a product of algebraic numbers is algebraic. ■

DEFINITION: A field extension $K \supset k$ is called algebraic if all elements of K are algebraic over k . A field k is called algebraically closed if all algebraic extensions of k are trivial.

EXAMPLE: The field \mathbb{C} is algebraically closed.

DEFINITION: In this lecture, k -algebra is a ring containing a field k , not necessarily with unity. All k -algebras are tacitly assumed commutative. Homomorphisms of k -algebras are k -linear maps compatible with the multiplication.

Minimal polynomials

CLAIM: Let K be a finite-dimensional k -algebra with unity and without zero divisors. **Then K is a field.**

Proof: An injective endomorphism of finite-dimensional spaces is surjective. Therefore, for each $x \in K$, there exists $y \in K$ such that $xy = 1$. ■

DEFINITION: Let v be an element of a finite-dimensional k -algebra R , and $P(t) = t^n + a_{n-1}t^{n-1} + \dots$ a polynomial of smallest possible degree with coefficients in k satisfying $P(v) = 0$. This polynomial is called **the minimal polynomial** of $v \in R$.

CLAIM: Let $v \in R$ be an element of finite-dimensional algebra R over k , and $P(t)$ its minimal polynomial. **Then the subalgebra $R_v \subset R$ generated by v is isomorphic to $k[t]/(P)$.**

Proof: By definition, R_v is a quotient of $k[t]$ by an ideal I of all polynomials $R(t)$ such that $R(v) = 0$. Since $k[t]$ is a principal ideal ring (home assignment 5), $I = (Q)$ for some polynomial $Q(t)$ satisfying $Q(v) = 0$. **Then Q is the minimal polynomial.** ■

Irreducible polynomials

THEOREM: The polynomial ring $k[t]$ is factorial (admits the unique prime decomposition).

Proof: See assignment 5. ■

DEFINITION: A polynomial $P(t) \in k[t]$ is **irreducible** if it is not a product of polynomials $P_1, P_2 \in k[t]$ of positive degree.

PROPOSITION: Let $(P) \subset k[t]$ be a principal ideal generated by the polynomial $P(t)$. Then **the polynomial $P(t)$ is irreducible if and only if the quotient ring $k[t]/(P)$ is a field.**

Proof. Step 1: The polynomial P is irreducible if and only if (P) is prime. This follows because $k[t]$ is a factorial ring.

Step 2: The quotient ring $k[t]/(P)$ is finite-dimensional over k . Then, it is a field if and only if it has no zero divisors. ■

Primitive extensions

DEFINITION: Let $P(t) \in k[t]$ be an irreducible polynomial. A field $k[t]/(P)$ is called **an extension of k obtained by adding a root of $P(t)$** . The extension $[k[t]/(P) : k]$ is called **primitive**.

CLAIM: Let $[K : k]$ be a finite extension. **Then K can be obtained from k by a finite chain of primitive extensions.** In other words, there exists a sequence of intermediate extensions $[K = K_n : K_{n-1} : K_{n-2} : \dots : K_0 = k]$ such that each $[K_i : K_{i-1}]$ is primitive. ■

Artinian algebras over a field

DEFINITION: A commutative, associative k -algebra R is called **Artinian algebra** if it is finite-dimensional as a vector space over k . Artinian algebra is called **semisimple** if it has no non-zero nilpotents.

DEFINITION: Let R_1, \dots, R_n be k -algebras. Consider their direct sum $\bigoplus R_i$ with the natural (term by term) multiplication and addition. This algebra is called **direct sum of R_i** , and denoted $\bigoplus R_i$.

Today we are going to prove the following theorem.

THEOREM: Let A be a semisimple Artinian ring. **Then A is a direct sum of fields, and this decomposition is uniquely defined.**

Idempotents

DEFINITION: Let $v \in R$ be an element of an algebra R satisfying $v^2 = v$. Then v is called **idempotent**.

REMARK: A product of two idempotents is clearly an idempotent. If e is an idempotent, then $1 - e$ is also an idempotent: $(1 - e)^2 = 1 - 2e + e^2 = 1 - e$.

COROLLARY: For each idempotent $e \in R$, one has $e(1 - e) = 0$. Therefore, **each idempotent $e \in A$ defines a decomposition of A into a direct sum: $A = eA \oplus (1 - e)A$.**

All Artinian algebras contain idempotents

THEOREM: Let A be an Artinian k -algebra without nilpotents. **Then A contains an idempotent.**

Proof. Step 1: Since A is finite-dimensional, every decreasing chain of ideals stabilizes. Therefore, **A contains an ideal $I \subset A$ which has no non-zero proper ideals.** We shall consider I as a sub-algebra in A .

Step 2: Since A has no nilpotents, for each non-zero $z \in I$ we have $z^2 \neq 0$. Since I is minimal, we have $zI = I$.

Step 3: Since I is finite-dimensional, **all elements of I are invertible as endomorphisms of I .**

Step 4: Since I is finite-dimensional, the elements $z, z^2, z^3, \dots \in \text{End } I$ are linearly dependent, which gives a polynomial relation $P(z) = 0$. If this polynomial has zero constant term, we divide it by z , and obtain another polynomial with the same property. Using induction, we obtain a polynomial relation $P(z) = 0$ with non-zero constant term. This gives a relation $\text{Id}_I = az + bz^2 + cz^3 + \dots$ in the ring $\text{End}_k(I)$, with $a, b, c, \dots \in k$.

Step 5: The element $U := az + bz^2 + cz^3 + \dots \in I$ satisfies $Ux = x$ for any $x \in I$. **Therefore, U is an idempotent in A , and unity in I . ■**

Structure theorem for semisimple Artinian algebras

REMARK: Step 5 proves the following useful statement. Let I be a commutative Artinian algebra without zero divisors. **Then I contains unit, that is, I is a field.**

COROLLARY: Let A be a semisimple Artinian ring, that is, a finite-dimensional commutative ring over k without nilpotents. **Then A is a direct sum of fields**

Proof: Let $I \subset A$ be a non-trivial ideal. As shown above, I contains a non-zero idempotent a . Then a and $b := 1 - a$ idempotents satisfying $ab = 0$, $a + b = 1$. **This gives a direct sum decomposition $A = aA \oplus (1 - a)A$.** Using induction in $\dim A$, we may assume already that aA and $(1 - a)A$ are direct sum of fields. ■

Structure theorem for semisimple Artinian algebras: uniqueness of decomposition

LEMMA: Let A be a direct sum of fields, $A = \bigoplus_i k_i$. **Then the decomposition $A = \bigoplus_i k_i$ is defined uniquely**, up to permutation of summands.

Proof: Let $A = \bigoplus_{i=1}^n k_i = \bigoplus_{j=1}^m k'_j$. and $a_1, \dots, a_n, b_1, \dots, b_n$ be the corresponding idempotents. Then the pairwise products $\{a_i b_j\}$ give a family of idempotents which satisfies $\sum a_i b_j = (\sum a_i) (\sum b_j) = 1$ and $a_i b_j a_{i'} b_{j'} = 0$ unless $i = i', j = j'$. Unless all idempotents $a_i b_j$ are equal to a_i , this gives a direct sum decomposition for each subfield k_i , which is impossible. Therefore, the sets $\{b_j\}$ and $\{a_i\}$ coincide. ■

Finite morphisms

REMARK: Let M be a finitely generated R -module, and $R \rightarrow R'$ a ring homomorphism. **Then $M \otimes_R R'$ is a finitely generated R' -module.** Indeed, **if M is generated by x_1, \dots, x_n , then $M \otimes_R R'$ is generated by x_1, \dots, x_n .**

DEFINITION: A morphism $X \rightarrow Y$ of affine varieties is called **finite** if the ring \mathcal{O}_X is a finitely generated module over \mathcal{O}_Y . In this case, \mathcal{O}_X is called **an integral extension** of \mathcal{O}_Y .

THEOREM: Let $X \xrightarrow{f} Y$ be a finite morphism. Then for any point $y \in Y$, **the preimage $f^{-1}(y)$ is finite.**

Proof. Step 1: Since \mathcal{O}_X is finite generated as an \mathcal{O}_Y -module, the ring $R := \mathcal{O}_X \otimes_{\mathcal{O}_Y} (\mathcal{O}_Y/\mathfrak{m}_y)$ is finitely generated as an $\mathcal{O}_Y/\mathfrak{m}_y$ -module. Since $\mathcal{O}_Y/\mathfrak{m}_y = \mathbb{C}$, we obtain that R is an Artinian algebra over \mathbb{C} .

Step 2: Let $N \subset R$ be a nilradical. As shown above, **$\text{Spec}(R/N)$ is a finite set.**

Step 3: On the other hand, as shown in the last lecture, $\text{Spec}(R/N) = f^{-1}(y)$. ■

Bilinear invariant forms

DEFINITION: Let R be a k -algebra, and $g : R \times R \rightarrow k$ a k -bilinear symmetric form on R . The form g is called **invariant** if $g(x, yz) = g(xy, z)$ for all $x, y, z \in R$.

REMARK: If R has unity, for any invariant form g we have $g(x, y) = h(xy, 1)$, hence g is determined by a linear functional $a \rightarrow g(a, 1)$.

EXAMPLE: Consider the ring $\mathbb{R}[x, y]/(x^{n+1}, y^{n+1})$, and let $\varepsilon\left(\sum a_{ij}x^i y^j\right) := a_{nn}$. **The corresponding bilinear invariant form $g(x, y) := \varepsilon(xy)$ is non-degenerate (prove this).**

CLAIM: Let $[K : k]$ be a field extension, and ε a non-zero k -linear functional on K . **Then the bilinear form $g(x, y) := \varepsilon(xy)$ is non-degenerate.**

Proof: Suppose $\varepsilon(a) \neq 0$. Then $g(x, x^{-1}a) \neq 0$. ■

The trace form

DEFINITION: Trace $\text{tr}(A)$ of a linear operator $A \in \text{End}_k(k^n)$ represented by a matrix (a_{ij}) is $\sum_{i=1}^n a_{ii}$.

DEFINITION: Let R be an Artinian algebra over k . Consider the bilinear form $a, b \rightarrow \text{tr}(ab)$, mapping a, b to the trace of endomorphism $L_{ab} \in \text{End}_k R$, where $l_{ab}(x) = abx$. This form is called **the trace form**, and denoted as $\text{tr}_k(ab)$.

REMARK: Let $[K : k]$ be a finite field extension. As shown above, **the trace form $\text{tr}_k(ab)$ is non-degenerate, unless tr_k is identically 0.**

Separable extensions

DEFINITION: A field extension $[K : k]$ is called **separable** if the trace form $\text{tr}_k(ab)$ is non-zero.

REMARK: If $\text{char } k = 0$, every field extension is separable, because $\text{tr}_k(1) = \dim_k K$.

THEOREM: Let R be an Artinian algebra over k with non-degenerate trace form. **Then R is semisimple.**

Proof: Since $\text{tr}_k(ab) = 0$ for any nilpotent a (indeed, the trace of a nilpotent operator vanishes), **the ring R contains no non-zero nilpotents.** ■

Tensor product of field extensions

LEMMA: Let R, R' be Artinian k -algebras. Denote the corresponding trace forms by g, g' . Consider the tensor product $R \otimes_k R'$ with a natural structure of Artinian k -algebra. **Then the trace form on $R \otimes_k R'$ is equal $g \otimes g'$,** that is,

$$\mathrm{tr}_{R \otimes_k R'}(x \otimes y, z \otimes t) = g(x, z)g'(y, t). \quad (*)$$

Proof: Let V, W be vector spaces over k , and μ, ρ endomorphisms of V, W . Then $\mathrm{tr}(\mu \otimes \rho) = \mathrm{tr}(\mu)\mathrm{tr}(\rho)$, which is clear from the block decomposition of the matrix $\mu \otimes \rho$. **This gives the trace for any decomposable vector $r \otimes r' \in R \otimes_k R'$.** The equation (*) is extended to the rest of $R \otimes_k R'$ by because decomposable vectors generate $R \otimes_k R'$. ■

COROLLARY: Let $[K_1 : k], [K_2 : k]$ be separable extensions. **Then the Artinian k -algebra $K_1 \otimes_k K_2$ is semisimple,** that is, isomorphic to a direct sum of fields.

Proof: The trace form on $K_1 \otimes_k K_2$ is non-degenerate, because $g \otimes g'$ is non-degenerate whenever g, g' is non-degenerate. ■

REMARK: In particular, **if $\mathrm{char} k = 0$, the product of finite extensions of the field k is always a direct sum of fields.**