Commutative Algebra

lecture 10: Normal varieties

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Zariski topology

DEFINITION: Zariski topology on an algebraic variety is a topology, where closed sets are algebraic subsets. **Zariski closure** of $Z \subset M$ is an intersection of all Zariski closed subsets containing Z.

DEFINITION: Cofinite topology is the topology on a set S such that the only closed subsets are S and finite sets.

EXERCISE: Prove that **Zariski topology on $\mathbb{C}$ coincides with the cofinite topology.**

CAUTION: Zariski topology is non-Hausdorff.

Zariski topology (2)

REMARK: We defined the Zariski topology on the set of points of A, that is, on the set of maximal ideals of $\mathcal{O}_A$ (this is how Zariski defined it). **Following Grothendieck, one defines the Zariski topology on the set $\operatorname{Spec}_{pr}(\mathcal{O}_A)$ of all prime ideals in $\mathcal{O}_A$: closed subsets Z_I in this topology correspond to prime ideals containing a given ideal $I \subset \mathcal{O}_A$.**

Oscar Zariski
(1899 – 1986)

Dominant morphisms

DEFINITION: **Dominant morphism** is a morphism $f: X \longrightarrow Y$, such that Y is a Zariski closure of $f(X)$.

PROPOSITION: Let $f: X \longrightarrow Y$ be a morphism of affine varieties. **The morphism f is dominant if and only if the homomorphism $\mathcal{O}_Y \xrightarrow{f^*} \mathcal{O}_X$ is injective**.

Proof. Step 1: If f^* is not injective, $f(X)$ lies in the set of common zeros of the ideal $\ker f^*$. Indeed, points of X are the same as maximal ideals and the same as homomorphisms $\mathcal{O}_X \longrightarrow \mathbb{C}$, and the points of $f(X)$ are homomorphisms $\mathcal{O}_Y \longrightarrow \mathbb{C}$ obtained as a composition $\mathcal{O}_Y \xrightarrow{f^*} \mathcal{O}_X \longrightarrow \mathbb{C}$.

Step 2: If $f(X)$ is contained in the set of common zeros of the ideal $J \subset \mathcal{O}_Y$, all functions $\alpha \in J$ vanish on $f(X)$. **This implies that $f^*(\alpha) = 0$.** ■

Field of fractions

DEFINITION: Let $S \subset R$ be a subset of R, closed under multiplication and not containing 0. **Localization** of R in S is a ring, formally generated by symbols a/F, where $a \in R$, $F \in S$, and relations $a/F \cdot b/G = ab/FG$, $a/F + b/G = \frac{aG+bF}{FG}$ and $aF^k/F^{k+n} = a/F^n$.

DEFINITION: Let R be a ring without zero divisors, and S the set of all non-zero elements in R. **Field of fractions** of R is a localization of R in S.

CLAIM: Let $f: X \longrightarrow Y$ be a dominant morphism, where X is irreducible. **Then Y is also irreducible**. Moreover, $f^*: \mathcal{O}_Y \longrightarrow \mathcal{O}_X$ **can be extended to a homomorphism of the fields of fractions $k(Y) \longrightarrow k(X)$.**

Proof. Step 1: Since $\mathcal{O}_Y$ is embedded in $\mathcal{O}_X$, and the later has no zero divisors, **$\mathcal{O}_Y$ has no zero divisors, hence Y is irreducible**.

Step 2: Embedding of rings without zero divisors can be extended to the fields of fractions: $f^*(a/F) = f^*(a)/f^*(F)$. ∎

Principal divisors

DEFINITION: Let X be an affine variety, and $f \in \mathcal{O}_X$ a regular function which does not vanish on any of irreducible components of X. The zero set of f is called **a principal divisor** on X. Its irreducible components are called **divisors** on X.

REMARK: Let $X \subset \mathbb{C}^n$ be an affine variety, given by ideal $I \subset \mathbb{C}[x_1, ..., x_n]$, and $D \subset X$ be a divisor. **Then $X \backslash D$ is an affine variety, given by an ideal $I + \langle ft - 1\rangle \subset \mathbb{C}[x_1, ..., x_n, t]$.**

DEFINITION: A dominant morphism of irreducible varieties is called **birational** if the corresponding homomorphism of the fields of fractions is an isomorphism.

EXAMPLE: **The natural map $X \backslash D \hookrightarrow X$ is birational.**

Birational morphisms

PROPOSITION: Let $f: X \longrightarrow Y$ be a birational morphism. **Then there exists a divisor $Z \subset Y$ such that $f: (X\backslash f^{-1}(Z)) \longrightarrow Y\backslash Z$ is an isomorphism.**

Proof. Step 1: Since $\mathcal{O}_X$ is finitely generated, **there exists $F \in \mathcal{O}_Y$ such that for all $a \in \mathcal{O}_X$ there exists $b \in \mathcal{O}_Y$ such that $a = f\left(\frac{b}{F^k}\right)$.** Indeed, for each generator x_i of $\mathcal{O}_X$ there exists $y_i, F_i \in \mathcal{O}_Y$ such that $x_i = f\left(\frac{y_i}{F_i}\right)$. Choosing $F = \prod F_i$, we obtain that $x_i = f\left(\frac{y_i'}{F}\right)$, where $y_i' = y_i \prod_{j\neq i} F_j$. Then for each homogeneous polynomial $P(x_i, ..., x_j)$ of degree d we have $P(x_1, ..., x_n) = f\left(\frac{P(y_1', ... y_n')}{F^d}\right)$.

Step 2: Let Z be the set of all $p \in Y$ such that $F(p) = 0$. **Then F is invertible in $Y\backslash Z$, hence $f^{-1}: Y\backslash Z \longrightarrow X$ is a polynomial map.** Therefore, $f: (X\backslash f^{-1}(Z)) \longrightarrow Y\backslash Z$ is invertible. ■

Integral dependence

DEFINITION: Let $A \subset B$ be rings. An element $b \in B$ is called **integral over A** if the subring $A[b] = A \cdot \langle 1, b, b^2, b^3, ... \rangle$, generated by b and A, is finitely generated as A-module.

DEFINITION: **Monic polynomial** is a polynomial with leading coefficient 1.

REMARK: When A is Noetherian, the following statement is an equivalence of two characterizations of Noetherian A-modules. However, it is true without the Noetherian assumtion.

CLAIM: An element $x \in B$ **is integral over $A \subset B$ if and only if the chain of submodules**

$$A \subset A \cdot \langle 1, x \rangle \subset A \cdot \langle 1, x, x^2 \rangle \subset A \cdot \langle 1, x, x^2, x^3 \rangle \subset ...$$

terminates.

Proof: If the chain terminates, then $A[x]$ is clearly finitely generated. Conversely, if $A[x]$ is finitely generated, any degree of x can be expressed through a finite number of generators, which can be expressed as polynomials on x. ■

COROLLARY: Let A be Noetherian. **An element $x \in B$ is integral over $A \subset B \Leftrightarrow x$ is a root of a monic polynomial with coefficients in A.** ■

Sum and product of integral elements is integral

EXERCISE: Let $x, y \in B \supset A$, with x integral over A and y integral over $A[x]$. Assume that A is Noetherian. **Prove that y is integral over A.**

CLAIM: Let $A \subset B$ be Noetherian rings. Then **sum and product of elements which are integral over A is also integral.**

Proof: Let $x, y \in B$ be integral over A. Since y is integral over $A[x]$, which is finitely generated as A-module, the ring $A[x, y]$ is finitely generated as an A-module. Since A is Noetherian, a submodule of finitely-generated A-module is finitely-generated, hence $x+y$ and $xy \in A[x, y]$ are also integral. ■

Integral closure

DEFINITION: Let $A \subset B$ be rings. The set of all elements in B which are integral over A is called **the integral closure of A in B**.

DEFINITION: Let A be the ring without zero divisors, and $k(A)$ its field of fractions. The set of all elements $a \in k(A)$ which are integral over A is called **the integral closure of A**. A ring A is called **integrally closed** if A coincides with its interal closure in $k(A)$.

REMARK: As shown above, **the integral closure is a ring.**

DEFINITION: An affine variety X is called **normal** if all its irreducible components X_i are disconnected, and the ring of functions $\mathcal{O}_{X_i}$ for each of these irreducible components is integrally closed.

Finite morphisms and normality

DEFINITION: (reminder from Lecture 8)
A morphism $X \longrightarrow Y$ of affine varieties is called **finite** if the ring $\mathcal{O}_X$ is a finitely generated module over $\mathcal{O}_Y$. In this case, $\mathcal{O}_X$ is called **an integral extension** of $\mathcal{O}_Y$.

PROPOSITION: X is normal **if and only if any finite, birational morphism $Y \longrightarrow X$ is an isomorphism.**

Proof: Since the fraction fields of Y and of X are equal, all elements of $\mathcal{O}_Y$ belong to the integral closure of X in the fraction field $k(X) = k(Y)$. This implies that $\mathcal{O}_Y = \mathcal{O}_X$. ∎

Factorial rings

DEFINITION: An element p of a ring R is called **irreducible** if for any decomposition $p = p_1 p_2$, either p_1 or p_2 is invertible.

DEFINITION: A ring R without zero divisors is called **factorial** if any element $r \in R$ can be represented as a product of irreducible elements, $r = \prod_i p_i^{\alpha_i}$, and this decomposition is unique up to invertible factors and permutation of p_i.

PROPOSITION: Let A be a factorial ring. Then it is integrally closed.

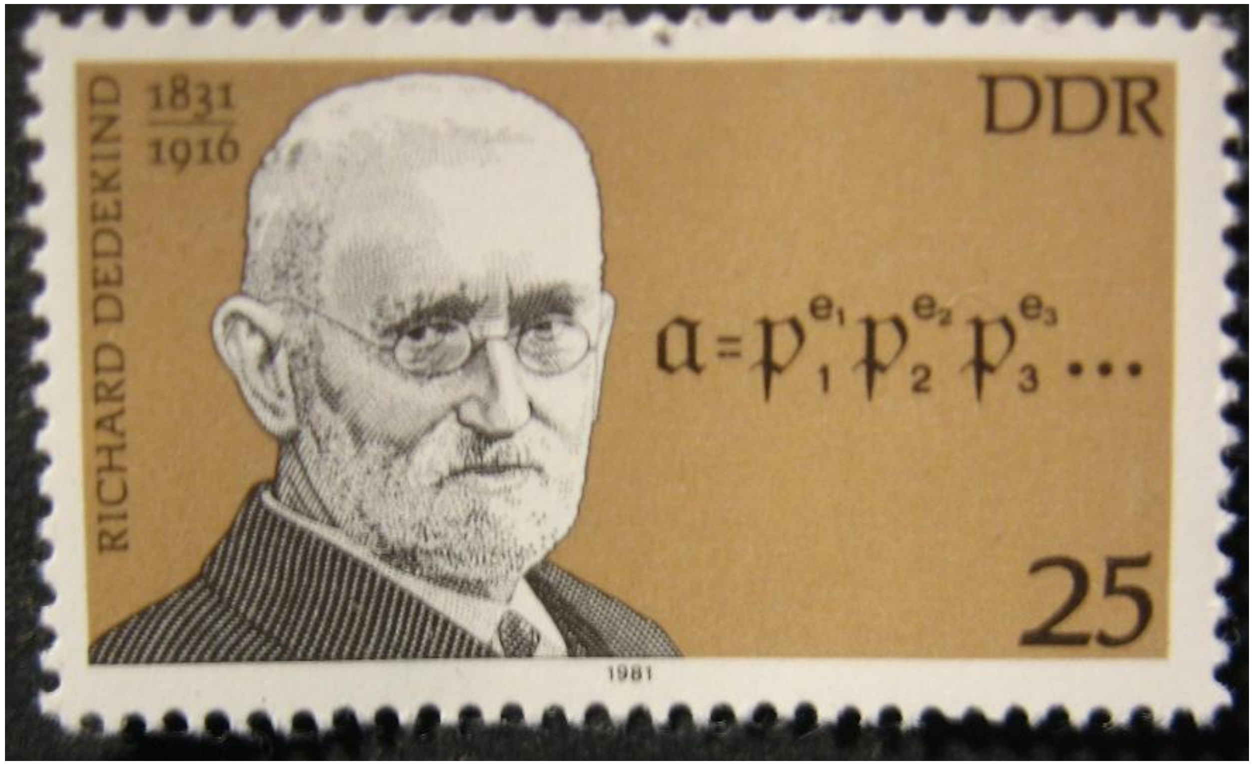
Proof. Step 1: Let $u, v \in A$, and $u/v \in k(A)$ a root of a monic polynomial $P(t) \in A[t]$ of degree n. **Then u^n is divisible by v in A.**

Step 2: Let $u/v \in k(A)$ be a root of a monic polynomial $P(t) \in A[t]$. Assume that u, v are comprime. **Since u^n is divisible by v, and they are coprime, v is invertible by factoriality of A.** Then $u/v \in A$. ■

Ernst Kummer

Ernst Eduard Kummer (1810-1893)

Richard Dedekind



Richard Dedekind (1831 - 1916)

Gauss lemma

EXERCISE: Let R be a ring without zero divisors. **Prove that the polynomial ring $R[t]$ has no zero divisors.**

THEOREM: ("Gauss lemma")
Let R be a factorial ring. **Then the ring of polynomials $R[t]$ is also factorial.**

Proof: See the next slide.

DEFINITION: Let R be a factorial ring. A polynomial $P(t) \in R[t]$ is called **primitive** if the greatest common divisor of its coefficients is 1.

Lemma 1: Let $P_1, P_2 \in R[t]$ be primitive polynomials. **Then their product is also primitive**.

Proof: Let $p \in R$ be a prime. Since the polynomials P_1, P_2 are primitive, they are non-zero modulo p. Since the ring $R/(p)$ has no zero divisors, **the product $P_1 P_2$ is non-zero in $R/(p)[t]$,** hence the greatest common divisor of the coefficients of $P_1 P_2$ is not divisible by p. ■

Irreducibility of polynomials in $R[t]$ and $K[t]$

Lemma 1: Let $P_1, P_2 \in R[t]$ be primitive polynomials. **Then their product is also primitive**.

Lemma 2: Let R be a factorial ring, and K its fraction field. **Then any primitive polynomial $P \in R[t]$, which is irreducible in $R[t]$, is also irreducible in $K[t]$.**

Proof: Assume that P is decomposable in $K[t]$. Then $rP = P_1P_2$, where $P_1, P_2 \in R[t]$ and $r \in R$. Let s_1, s_2 be the greatest common divisors of the coefficients of P_1, P_2. Then $rP = s_1s_2P_1'P_2'$, and P_1', P_2' are primitive. In this case $P_1'P_2'$ is primitive (Lemma 1), hence the greatest common divisor of the coefficients of $s_1s_2P_1'P_2'$ is s_1s_2. Since P is also primitive, the greatest common divisor of the coefficients of $rP = s_1s_2P_1'P_2'$ is r. **Then $\frac{r}{s_1s_2}$ is invertible, and P is decomposable in $R[t]$.** ■

Gauss lemma (proof)

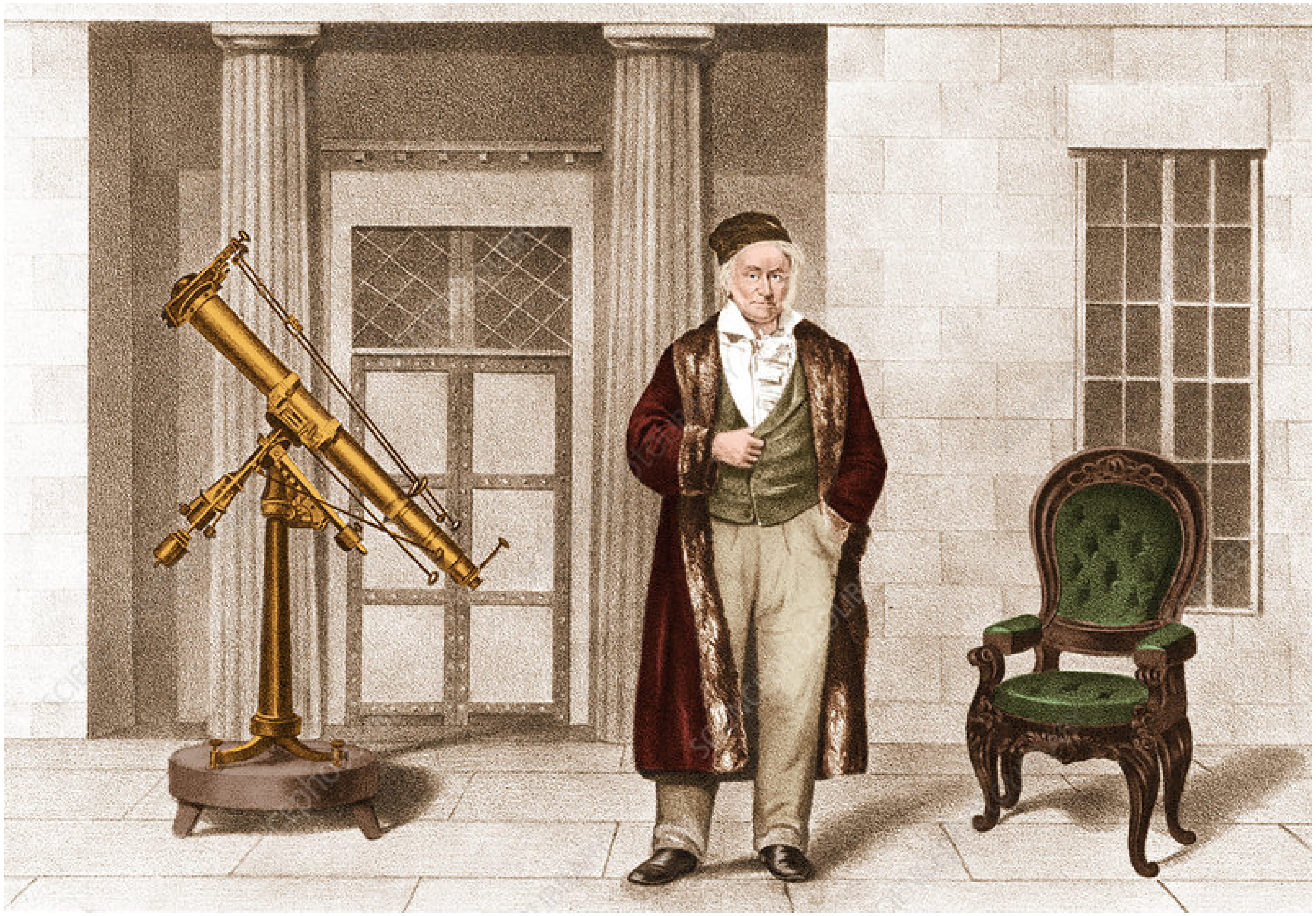
THEOREM: ("Gauss lemma")
Let R be a factorial ring. **Then the ring of polynomials $R[t]$ is also factorial.**

Proof: Let K be the fraction field of R. The ring $K[t]$ is factorial, because it is Euclidean (handout 3). Lemma 2 implies that a prime decomposition of a primitive polynomial $P(t) \in R[t]$ is uniquely determined by its prime decomposition in $K[t]$, hence it is unique. A non-primitive polynomial is decomposed as a product of the greatest common divisor of its coefficients and a primitive polynomial, hence its prime decomposition is also unique. ■

COROLLARY: The affine space $\mathbb{C}^n$ is a normal variety. Moreover, **for any variety X with factorial ring $\mathcal{O}_X$ of regular functions, the product $X \times \mathbb{C}^n$ is also normal.**

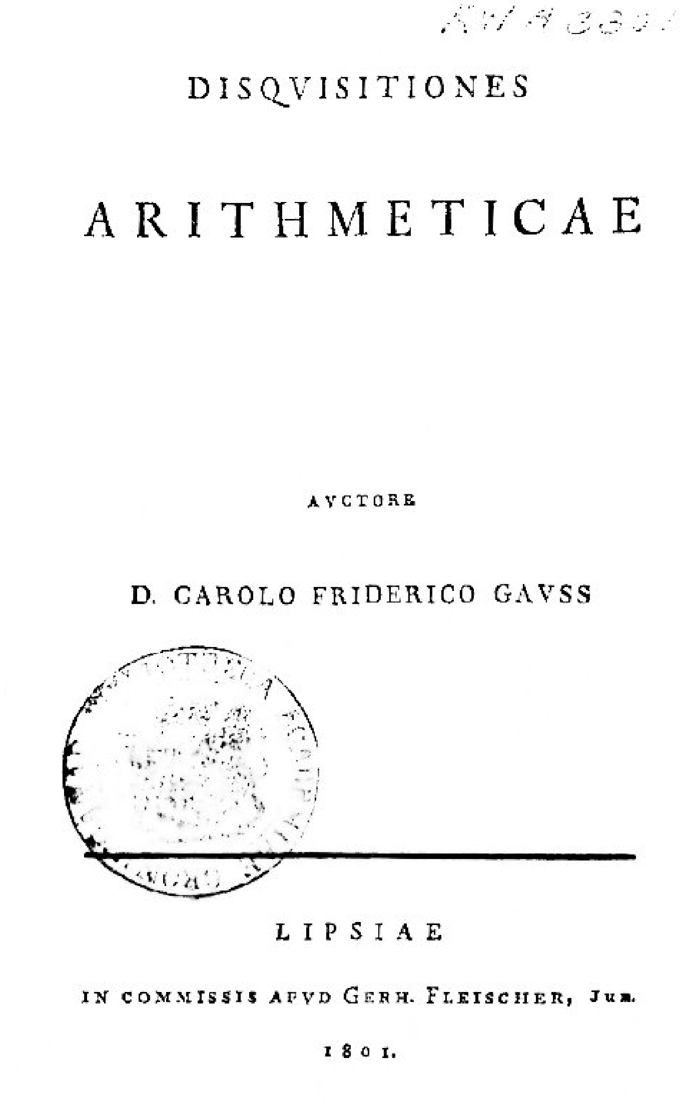
Proof: As we have shown previously, $\mathcal{O}_{X\times\mathbb{C}^n} = \mathcal{O}_X \otimes_{\mathbb{C}} \mathbb{C}[t_1, ..., t_n] = \mathcal{O}_X[t_1, ..., t_n]$. This ring is factorial by Gauss lemma. ■

Carl Friedrich Gauss



Carl Friedrich Gauss (1777 - 1855)

Disquisitiones Arithmeticae



DISQVISITIONES

ARITHMETICAE

AVCTORE

D. CAROLO FRIDERICO GAVSS

LIPSIAE

IN COMMISSIS APVD GERH. FLEISCHER, IVN.

1801.

“Disquisitiones Arithmeticae”,
written by Gauss in 1798, in Latin, when he was 21.
This book contains “Gauss Lemma”.