

Commutative Algebra

lecture 11: Nakayama's lemma

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Nakayama's lemma

QUESTION: Let $\mathfrak{a} \subset A$ be a non-trivial ideal in a Noetherian ring. **How can we prove that $\bigcap_i \mathfrak{a}^i = 0$?**

ANSWER: Nakayama's lemma!

REMARK: $\bigcap_i \mathfrak{a}^i = 0$ **does not hold** in the ring of smooth functions, which is non-Noetherian.

DEFINITION: An A -module M is called **torsion-free** if for any non-zero $a \in A$, and any non-zero $m \in M$, one has $am \neq 0$.

Nakayama's lemma (also known as "Krull-Azumaya theorem"): Let M a finitely-generated torsion-free A -module. **Then for any non-trivial ideal $\mathfrak{a} \subset A$, $\mathfrak{a}M = M$ implies $M = 0$.**



Tadashi Nakayama
(1912-1964)

The Jacobson ideal

DEFINITION: Jacobson ideal in a ring is the intersection of all maximal ideals.

Claim 1: An element $a \in R$ belongs to the Jacobson ideal $J(A)$ **iff $1 - ra$ is invertible for any $r \in R$.**

Proof. Step 1: If a does not belong to a maximal ideal \mathfrak{m} , $1 = ar \pmod{\mathfrak{m}}$ for some r , hence $1 - ar$ cannot be invertible. Therefore, **invertibility of $1 - ra$ for all r implies $a \in J(R)$.**

Step 2: Conversely, suppose that $1 - ra$ is not invertible for some $r \in R$. Then $1 - ra \in \mathfrak{m}$ for some maximal ideal $\mathfrak{m} \subset R$. If $a \in J(R)$, this implies that $1 - ra \in \mathfrak{m}$ and $ra \in \mathfrak{m}$, hence $1 \in \mathfrak{m}$, which is impossible. Therefore, **for any $a \in J(R)$, $1 - ra$ is invertible for any $r \in R$.** ■

EXERCISE: Let \mathcal{O}_A be a finitely generated ring over \mathbb{C} without zero divisors. **Find its Jacobson ideal.**

Quasi-regular ideals

DEFINITION: An ideal $\mathfrak{a} \subset R$ is **quasi-regular** if $1 + a$ is invertible for any $a \in \mathfrak{a}$.

PROPOSITION: An ideal is quasi-regular **if and only if it belongs to the Jacobson ideal.**

Proof: If $\mathfrak{a} \subset J(R)$, this implies $1 + ra$ is invertible for all $r \in R$ (Claim 1), in particular $1 + a$ is invertible. Conversely, if $\mathfrak{a} \subset R$ is a quasi-regular ideal, then ra is quasi-regular for all r , hence $1 + ra$ is invertible. ■

REMARK: Let $a, b \in \mathfrak{a}$. **Then $ab \in \mathfrak{a}$,** hence the set S of such elements is multiplicative. Therefore, **we can localize the ring R in the set of all such elements. The localization $R(S^{-1})$ is a quasi-regular ring,** because for any $a, b \in \mathfrak{a}$, the element $z := 1 + (1 + a)^{-1}b$ satisfies $(1 + a)z = (1 + a) + b$, and $(1 + a) + b$ is invertible.

Localization of modules

DEFINITION: Let $S \subset A$ be a multiplicative system, and $A(S^{-1})$ the localization of A in S . **Localization of A -module M in S** is $M(S^{-1}) := M \otimes_A A(S^{-1})$.

Claim 2: Consider the natural map $\tau : M \rightarrow M(S^{-1})$ taking m to $m \otimes_A 1$. **Then $\tau(x) = 0$ if and only if $sx = 0$ for some $s \in S$.**

Proof: The ring $A(S^{-1})$ is the quotient of a ring freely generated by the “fractions” $\frac{a}{s}$, $a \in A, s \in S$, subject to three relations $\frac{a}{s} + \frac{b}{s} = \frac{a+b}{s}$, $\frac{a}{s} \frac{b}{s'} = \frac{ab}{ss'}$ and $\frac{as'}{s's} = \frac{a}{s}$. Therefore, $M(S^{-1})$ is the quotient of a module freely generated by the fractions $\frac{m}{s}$, $m \in M, s \in S$, and subject to the relations $\frac{m}{s} + \frac{m'}{s} = \frac{m+m'}{s}$, $\frac{am}{s s'} = \frac{am}{s s'}$ and $\frac{s'm}{s's} = \frac{m}{s}$. The first two relations don't affect the injectivity of the tautological map from M : the corresponding module is identified with the sum $\bigoplus_{s \in S} s^{-1}M$, considered as a module over $\bigoplus_{s \in S} s^{-1}A$, and τ is the embedding of M to the component $s_0^{-1}M$ for $s_0 = 1$. The kernel of the map $\tau : M \rightarrow M(S^{-1})$ is obtained from the third relation, hence it is generated by all $m \in M$ such that $sm = 0$ for some $s \in M$. ■

Nakayama's lemma and quasi-regular ideals

Nakayama, 1951: Let M a finitely-generated A -module. **Then for any non-trivial ideal $\mathfrak{a} \subset A$, $\mathfrak{a}M = M$ implies that $(1 + a)M = 0$,** for some $a \in \mathfrak{a}$ is a non-zero element.

CLAIM: Nakayama's lemma is equivalent to the following assertion.

Let M a finitely-generated A -module, and $\mathfrak{a} \subset A$ a quasiregular ideal. Then $\mathfrak{a}M = M$ implies $M = 0$.

Proof: Nakayama's lemma implies this statement directly, because $(1+a)M = 0$, for invertible $1 + a$, implies $M = 0$. To prove the converse, consider the quasi-regular ring A_q obtained from A by localization in all elements $1 + a$ satisfying $a \in \mathfrak{a}$, and let $M_q := M \otimes_A A_q$. Since $\mathfrak{a}M = M$ and A_q is quasi-regular, this implies that $M_q = 0$. Let x_1, \dots, x_n be generators of M . By Claim 2, $M_q = 0$ implies that there exist $s_1, \dots, s_n \in 1 + \mathfrak{a}$ such that $s_i m_i = 0$, hence $\prod_i s_i M = 0$. Since $\prod_i s_i \in 1 + \mathfrak{a}$, this implies the Nakayama's lemma. ■

Nakayama's lemma for modules over quasi-regular rings

We are proving Nakayama's lemma in the following form.

PROPOSITION: Let M a finitely-generated A -module, and $\mathfrak{a} \subset A$ a quasiregular ideal. **Then $\mathfrak{a}M = M$ implies $M = 0$.**

Proof. Step 1: We use induction in the number of generators. Assume first that M is generated by just one element z (in this case we say that M is a **cyclic A -module**). Then $z = az$, where $a \in A$, which implies $(1 - a)z = 0$. **Since $1 - a$ is invertible, this means that $M = 0$.**

Step 2: Let z_1, \dots, z_n be generators of M . Then $z_i = \sum_{j=1}^n a_{ij}z_j$, with $a_{ij} \in \mathfrak{a}$, which gives $(1 - a_{11})z_1 = \sum_{j=2}^n a_{1j}z_j$. Since \mathfrak{a} is quasiregular, $1 - a_{11}$ is invertible, hence, after replacing each a_{1j} by $a'_{1j} := (1 - a_{11})^{-1}a_{1j}$, we can assume $z_1 = \sum_{j=2}^n a'_{1j}z_j$. Substituting $z_1 = \sum_{j=2}^n a'_{1j}z_j$ into $z_i = \sum_{j=1}^n a_{ij}z_j$ for $i = 2, 3, \dots, n$, we obtain that $z_i = \sum_{j=2}^n a'_{ij}z_j$, where $i = 2, 3, \dots, n$ and $a'_{ij} \in \mathfrak{a}$. **This gives a submodule $M_1 \subset M$ generated by $n - 1$ generators z_2, \dots, z_n and satisfying $\mathfrak{a}M_1 = M_1$.**

Step 3: By our induction assumption, the statement of the proposition holds for M_1 , which gives $M_1 = 0$, hence M is cyclic, and $M = 0$ by Step 1. ■

Krull theorem

Nakayama's lemma for torsion-free A -modules: Let M be a finitely-generated torsion-free A -module. **Then for any non-trivial ideal $\mathfrak{a} \subset A$, $\mathfrak{a}M = M$ implies $M = 0$.**

Proof: $(1 + \mathfrak{a})M = 0$ implies $M = 0$, because M is torsion-free. ■

THEOREM: (Krull theorem)

Let $\mathfrak{a} \subset A$ be an ideal in a Noetherian ring without zero divisors. Then $\bigcap \mathfrak{a}^n = 0$.

Proof: Let $M := \bigcap \mathfrak{a}^n$. This is a torsion-free module satisfying $\mathfrak{a}M = M$. Since A is Noetherian, M is finitely generated. Nakayama's lemma implies $M = 0$. ■

REMARK: Krull and Azumaya **independently proved the Nakayama lemma for ideals in a ring.**



Wolfgang Krull (1899-1971), Göttingen, 1920

Local rings

DEFINITION: A ring A is called **local** if it has only one maximal ideal.

DEFINITION: Let $\mathfrak{p} \subset A$ be a prime ideal, and $S \subset A$ its complement. **Localization of A in \mathfrak{p}** is $A[S^{-1}]$.

CLAIM: Localization of A in \mathfrak{p} is local.

Proof: Any $x \in A \setminus \mathfrak{p}$ is invertible, hence A/\mathfrak{p} is a field. Therefore, \mathfrak{p} is **maximal ideal**. It contains all ideals in A , because any element in $A \setminus \mathfrak{p}$ is invertible. ■

CLAIM: Let A be a ring, \mathfrak{m} its maximal ideal, and $\Phi \in \text{Hom}_A(M_1, M_2)$ a homomorphism of finitely-generated A -modules. Suppose that Φ induces a surjective map $\text{Hom}_{A/\mathfrak{m}}(M_1/\mathfrak{m}M_1, M_2/\mathfrak{m}M_2)$. **Then Φ is surjective.**

Proof: Let $M_3 := \text{coker } \Phi$. For any $x \in M_2$, one has $x \in \text{im } \Phi \pmod{\mathfrak{m}}$. Therefore, $\mathfrak{m}M_3 = M_3$. Then Nakayama's lemma implies that $(1 + a)M_3 = 0$, for some $a \in \mathfrak{m}$. Since $1 + a$ is invertible, this implies that $M_3 = 0$. ■

Finite morphisms (reminder)

REMARK: Let M be a finitely generated R -module, and $R \rightarrow R'$ a ring homomorphism. **Then $M \otimes_R R'$ is a finitely generated R' -module.** Indeed, **if M is generated by x_1, \dots, x_n , then $M \otimes_R R'$ is generated by $x_1 \otimes 1, \dots, x_n \otimes 1$.**

DEFINITION: A morphism $X \rightarrow Y$ of affine varieties is called **finite** if the ring \mathcal{O}_X is a finitely generated module over \mathcal{O}_Y . In this case, \mathcal{O}_X is called **an integral extension** of \mathcal{O}_Y .

THEOREM: Let $X \xrightarrow{f} Y$ be a finite morphism. Then for any point $y \in Y$, **the preimage $f^{-1}(y)$ is finite.**

Proof. Step 1: Since \mathcal{O}_X is finite generated as an \mathcal{O}_Y -module, the ring $R := \mathcal{O}_X \otimes_{\mathcal{O}_Y} (\mathcal{O}_Y/\mathfrak{m}_y)$ is finitely generated as an $\mathcal{O}_Y/\mathfrak{m}_y$ -module. Since $\mathcal{O}_Y/\mathfrak{m}_y = \mathbb{C}$, we obtain that R is an Artinian algebra over \mathbb{C} .

Step 2: Let $N \subset R$ be a nilradical. As shown in Lecture 10, **$\text{Spec}(R/N)$ is a finite set.**

Step 3: As shown in Lecture 8, **$\text{Spec}(R/N) = f^{-1}(y)$.** ■

Dominant, finite morphisms are surjective

THEOREM: Let $f : X \rightarrow Y$ be a finite, dominant morphism of affine varieties. **Then f is surjective.**

Proof. Step 1: Restricting to irreducible components, we can always assume that Y , and hence X is irreducible. Let $A = \mathcal{O}_Y$, $B = \mathcal{O}_X$. We can consider A as a subring of B , which has no zero divisors, and assume that B is finitely generated as an A -module.

Step 2: Let $\mathfrak{m}_y \subset A$ be a maximal ideal corresponding to $y \in Y$. **Nakayama's lemma implies that $\mathfrak{m}_y B \neq B$.**

Step 3: $f^{-1}(y) = \text{Spec}(B \otimes_A A/\mathfrak{m}_y) = \text{Spec}(B/\mathfrak{m}_y B)$. **Since this is non-zero ring, the set $f^{-1}(y)$ is non-empty. ■**

Projective modules over local rings

DEFINITION: Recall that a **projective A -module** is a direct summand of the free module A^n .

REMARK: Equivalently, P is a projective A -module **if and only if for any surjective map $M \rightarrow N$ of A -modules, the corresponding map $\text{Hom}(P, M) \rightarrow \text{Hom}(P, N)$ is surjective.**

THEOREM: Let P be a finitely generated projective module over a local ring (A, \mathfrak{m}) . **Then P is free.**

Proof. Step 1: Let $p_1, \dots, p_n \in P$ be a collection of elements which, after passing to $P/\mathfrak{m}P$, gives a basis. **The corresponding map $\Phi : A^n \rightarrow P$ is surjective by Nakayama lemma.**

Step 2: The map $\text{Hom}(P, A^n) \rightarrow \text{Hom}(P, P)$ induced by Φ is surjective, hence there is a map $u \in \text{Hom}(P, A^n)$ such that $u \circ \Phi = \text{Id}_P$. Therefore, **the map $\Phi : A^n \rightarrow P$ admits a section s .**

Step 3: By construction, $A^n/\mathfrak{m} = P/\mathfrak{m}$, **hence s is also surjective by Nakayama lemma;** since its composition with Φ is identity, s is an isomorphism. ■

Grothendieck group for local rings

DEFINITION: Given a ring R , let **the Grothendieck group** $K_0(R)$ be the group generated by isomorphism classes of finitely generated projective R -modules up to the equivalence $[P_1 \oplus P_2] = [P_1] \oplus [P_2]$.

REMARK: Since a tensor product of projective modules is projective, **the Grothendieck group** $K_0(R)$ **is actually a ring.**

PROPOSITION: Let (A, \mathfrak{m}) be a local ring. **Then** $K_0(A) = \mathbb{Z}$.

Proof: Let $\chi : K_0(A) \rightarrow \mathbb{Z}$ take a module P to $\dim_{A/\mathfrak{m}} P/\mathfrak{m}P$. **Since all projective A -modules are free, this map is an isomorphism. ■**