

# Commutative Algebra

## lecture 12: Normalization

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## Transcendental extensions

**CLAIM:** Let  $[k : \mathbb{C}]$  be an extension of  $\mathbb{C}$ , and  $[K : k]$  an extension of  $k$  generated over  $k$  by  $z \in K$ . **Then either  $z$  is transcendental, and  $K$  is isomorphic to the field of rational functions  $k(z)$ , or  $z$  is algebraic, and  $[K : k]$  is a finite extension.**

**Proof:** Indeed, either  $z$  is a root of polynomial, and then  $[K : k]$  is finite, or  $K$  contains the polynomial ring  $k[z]$ , and then  $K$  contains  $k(z)$ . ■

**LEMMA:** Let  $[K : k]$  be a finite extension, and  $[K(t) : k(t)]$  the corresponding extension of rational functions. **Then  $[K(t) : k(t)]$  is also finite.**

**Proof:** Primitive element theorem gives  $K = k[\alpha]$ , where  $\alpha$  is an algebraic element, which is a root of a polynomial  $P(z)$ . Using the isomorphism  $A/J \otimes_A B = B/JB$ , we obtain

$$K(t) = K \otimes_k k(t) = \frac{k[z]}{(P)} \otimes_k k(t) = k(t)[z]/(P).$$

■

## Transcendence basis

**DEFINITION:** Let  $k(t_1, \dots, t_n)$  be the field of rational functions of several variables, that is, the fraction field for the polynomial ring  $k[t_1, \dots, t_n]$ . Then the extension  $[k(t_1, \dots, t_n) : k]$  is called **a purely transcendental extension of  $k$** , and  $t_1, \dots, t_n$  are called **algebraically independent**.

**REMARK:** Clearly,  $t_1, \dots, t_n$  are algebraically independent if and only if there are no algebraic relations of form  $P(t_1, \dots, t_n) = 0$ , where  $P$  is a polynomial of  $n$  variables.

**DEFINITION:** **Transcendence basis** of an extension  $[K : k]$  is a collection  $z_1, \dots, z_n \in K$  generating a purely transcendental extension  $K' := k(z_1, \dots, z_n)$  such that  $[K : K']$  is an algebraic extension.

## Transcendence basis in regular functions

**Theorem 1:** Let  $k(X)$  be the field of rational functions on an irreducible affine variety  $X$ , with  $\mathcal{O}_X$  generated by  $t_1, \dots, t_n$ . Let  $S \subset \{t_1, \dots, t_n\}$  be a maximal algebraically independent subset. **Then the extension  $[k(X) : k(t_1, \dots, t_k)]$  is finite.**

**Proof:** Since  $\mathcal{O}_X$  is finitely generated, we can use induction by the number of generators  $t_1, \dots, t_n$  of  $\mathcal{O}_X$ . Let  $A \subset \mathcal{O}_X$  be a subring generated by  $t_1, \dots, t_{n-1}$ , and  $t_1, \dots, t_k$  a transcendence basis on  $k(A)$ .

If  $t_n$  is algebraic over  $k(A)$ , then  $[k(X) : k(A)]$  is finite; since  $[k(A) : k(t_1, \dots, t_k)]$  is finite, this implies that  $[k(X) : k(t_1, \dots, t_k)]$  is finite.

If  $t_n$  is transcendental over  $k(A)$ , we obtain  $k(X) = k(A)(t_n)$ , and  $[k(X) : k(t_1, \dots, t_k)] = [k(A)(t_n) : k(t_1, \dots, t_k, t_n)]$  is finite over  $k(t_1, \dots, t_k, t_n)$  by the lemma above. ■

## Transcendence basis and dominant morphisms

**PROPOSITION:** Let  $X \subset \mathbb{C}^n$  be an irreducible affine manifold,  $t_1, \dots, t_n$  coordinates on  $\mathbb{C}^n$ , and  $\Pi_k : X \rightarrow \mathbb{C}^k$  the projection to the first  $k$  coordinates.

**Then the following are equivalent.**

**(i)  $\Pi_k$  is dominant and the extension  $[k(X) : k(t_1, \dots, t_k)]$  is finite.**

**(ii)  $t_1, \dots, t_k$  is transcendence basis in  $k(X)$ .**

**Proof:** Theorem 1 implies that  $[k(X) : k(t_1, \dots, t_k)]$  is finite whenever  $t_1, \dots, t_k$  is the transcendence basis. Therefore, (ii)  $\Rightarrow$  (i). Converse is clear, because  $k(X)$  is algebraic over  $k(t_1, \dots, t_k) \subset k(X)$ , hence  $t_1, \dots, t_k$  is a maximal algebraically independent subset. ■

**REMARK:** We want to find a projection  $\Pi_k : X \rightarrow \mathbb{C}^k$  which is finite, for any given irreducible affine variety  $X$ .

## When the coordinate projection is finite

**REMARK:** Let  $X \subset \mathbb{C}^n$  be an irreducible affine subvariety,  $z_i$  coordinates on  $\mathbb{C}^n$ , and  $z_1, \dots, z_k$  transcendence basis on  $k(X)$ . **The projection map  $\Pi_{n-1}$  is finite if and only if  $P(z_n) = 0$  in  $\mathcal{O}_X$ , for some monic polynomial  $P(t) \in \mathcal{O}_X[t]$  with coefficients which are polynomial in  $z_1, \dots, z_{n-1}$ .** Indeed, this is precisely what is needed for  $\mathcal{O}_X$  to be a finitely generated module over its subalgebra  $A = \mathcal{O}_{\Pi_{n-1}(X)}$  generated by  $z_1, \dots, z_{n-1}$  considered as regular functions on  $X$ . Notice that **a non-zero polynomial with  $P(t) \in A[t]$  with  $P(z_n) = 0$  always exists, unless  $n = k$  and  $X = \mathbb{C}^n$ ,** but it is not necessarily monic.

**CLAIM:** In these assumptions, **there exists a linear coordinate change  $z'_i := z_i + \lambda_i z_n$ , such that  $z_n$  is finite over  $z'_1, \dots, z'_k$ ,** that is,  $z_n$  is a root of monic polynomial with coefficients in  $\mathbb{C}[z'_1, \dots, z'_k]$ .

**Proof:** Next page

## When the coordinate projection is finite (2)

**CLAIM:** In these assumptions, **there exists a linear coordinate change**  $z'_i := z_i + \lambda_i z_n$ , **such that**  $z_n$  **is finite over**  $z'_1, \dots, z'_k$ .

**Proof. Step 1:** Let  $P(z_1, \dots, z_k, z_n) = 0$  be a non-zero polynomial relation in  $\mathcal{O}_X$ , of positive degree in  $z_n$ . Such a relation exists because  $z_1, \dots, z_k$  is a transcendence basis in  $k(X)$ , and  $z_n$  is algebraic over  $z_1, \dots, z_k \in \mathcal{O}_X$ . Choose  $P$  which has minimal possible degree in  $z_1, \dots, z_k, z_n$ . Let  $F_P(z_1, \dots, z_k, z_n)$  be a homogeneous component of top degree  $d$  in  $P(z_1, \dots, z_k, z_n)$ .

**Step 2:** Consider a polynomial

$$Q(z'_1, \dots, z'_k, z_n) := F(z_1 + \lambda_1 z_n, \dots, z_k + \lambda_k z_n, z_n).$$

**Then**  $Q(0, 0, \dots, 0, 1) = F(\lambda_1, \dots, \lambda_k, 1)$  **is non-zero on**  $X$  **for general**  $\lambda_i$ . Indeed, if  $F(\lambda_1, \dots, \lambda_k, 1)$  is identically 0 for all  $\lambda_i$ , the top degree homogeneous term in  $P(z_1, \dots, z_k, z_n)$  vanishes on  $X$ , and we can replace  $P(z_1, \dots, z_k, z_n)$  by a smaller degree polynomial.

**Step 3:** The polynomial  $Q(z'_1, \dots, z'_k, t)$  is homogeneous of degree  $d$ , which is the maximal degree of all terms in  $P(z_1, \dots, z_k, z_n)$ . The degree  $d$  polynomial  $P(z'_1, \dots, z'_k, t)$  is monic in  $t$ , because its leading term  $t^d$  has non-zero scalar coefficient by Step 2, hence  $z_n$  **is finite over**  $z'_1, \dots, z'_k$ . ■

## Noether's normalization lemma

**REMARK:** Let  $A \subset B \subset C$  be ring without zero divisors,  $C$  is finitely generated as  $B$ -module, and  $B$  as an  $A$ -module. Then  $C$  is finitely generated as an  $A$ -module (Lecture 10).

**REMARK:** We have proven that the projection  $\Pi_{n-1} : X \rightarrow \mathbb{C}^{n-1}$  to coordinates  $z'_1, \dots, z'_k, z_{k+1}, \dots, z_{n-1}$  is finite. Using this remark and induction by  $n$ , we obtain

### **COROLLARY: (Noether's normalization lemma)**

Let  $X \subset \mathbb{C}^n$  be an irreducible affine subvariety. Then there exists a linear coordinate change such that **the projection of  $X$  to the first  $k$  coordinates gives a finite map  $X \rightarrow \mathbb{C}^k$ .** ■

**REMARK:** This is expressed by saying that **the projection of  $X$  to the first  $k$  coordinates is a branched cover.**

## Noether's normalization lemma (2)

**Corollary 1:** Let  $X \subset \mathbb{C}^n$  be an irreducible affine subvariety,  $z_i$  coordinates on  $\mathbb{C}^n$ , and  $z_1, \dots, z_k$  transcendence basis on  $k(X)$ . Then **there exists a linear coordinate change  $z'_i := z_i + \sum_{j=1}^{n-k} \lambda_{j+k} z_{j+k}$ , such that the projection  $\Pi_k : X \longrightarrow \mathbb{C}^k$  to the first  $k$  arguments is a finite, dominant morphism.**

**Proof:** Previous proposition shows that the projection  $P_n : X \longrightarrow \mathbb{C}^{n-1}$  is finite onto its image  $X_1$  (after some linear adjustment). Using induction by  $n$ , we can assume that  $P_k : X_1 \longrightarrow \mathbb{C}^k$  is also finite, hence the composition map is finite **(composition of finite morphisms is always finite, as we have seen)**. ■

## Noether's normalization lemma for non-irreducible varieties

The following version works for non-irreducible varieties.

**PROPOSITION:** Let  $X \subset \mathbb{C}^n$  be an affine subvariety, and  $X_i$  its irreducible components. Denote by  $k$  the maximal transcendence degree for  $k(X_i)$ . Then **there exists a linear coordinate change  $z'_i := z_i + \sum_{j=1}^{n-k} \lambda_{j+k} z_{j+k}$ , such that the projection  $\Pi_k : X \rightarrow \mathbb{C}^k$  to the first  $k$  arguments is a finite.**

**Proof. Step 1:** The natural projection map

$$\psi : \mathcal{O}_X \longrightarrow \prod_{\mathfrak{m} \in \text{Spec}(\mathcal{O}_X)} \mathcal{O}_X/\mathfrak{m}$$

is injective by Hilbert Nullstellensatz (all functions in its kernel vanish on  $X$ ).

**Step 2:** The natural projection map  $\phi : \mathcal{O}_X \rightarrow \bigoplus \mathcal{O}_{X_i}$  is injective, because  $\psi$  factorizes through  $\phi$ . It is also finite, because  $\mathcal{O}_{X_i}$  is finitely generated over  $\mathcal{O}_X$ . Clearly,  $\coprod X_i = \text{Spec}(\bigoplus \mathcal{O}_{X_i})$ , where  $\coprod$  denotes the disjoint union.

**Step 3:** Choose a coordinate projection  $\Pi_k : \mathbb{C}^n \rightarrow \mathbb{C}^k$  which is finite on each  $X_i$ ; such a projection exists by Corollary 1. The composition  $\coprod X_i \rightarrow X \xrightarrow{\Pi_k} \mathbb{C}^k$  is finite, hence  $\bigoplus \mathcal{O}_{X_i}$  is a finitely generated  $\mathcal{O}_{\mathbb{C}^k}$ -module. Since  $\mathcal{O}_{\mathbb{C}^k}$  is Noetherian, the submodule  $\mathcal{O}_X \subset \bigoplus \mathcal{O}_{X_i}$  is also finitely generated. ■



## Emmy Noether (1882-1935)



*Emmy Noether (1882-1935)*

## Finiteness of integral closure for Noetherian rings

**DEFINITION:** Recall that the ring  $B \subset K \supset A$  of all elements  $x \in K$  such that  $A[x] \subset K$  is a finitely-generated  $A$ -module is called **the integral closure of  $A$  in  $K$** . **The integral closure** of  $A$  is its integral closure in the field of fractions  $k(A)$ .

**Theorem 1:** Let  $A$  be an integrally closed Noetherian ring without zero divisors,  $[K : k(A)]$  a finite extension of its field of fractions, and  $B$  the integral closure of  $A$  in  $K$ . **Then  $B$  is finitely generated as an  $A$ -module.**

**Proof. Step 1:** For any  $b \in B$ , denote by  $L_b : K \rightarrow K$  the map of multiplication by  $b$ . Consider  $L_b$  as a  $k(A)$ -linear endomorphism of the finite-dimensional space  $K$  over  $k(A)$ , and define **the trace**  $\text{Tr}(b) := \text{Tr}(L_b)$ . Clearly,  $\text{Tr}(b) = \frac{d}{dt} \det(t\text{Id}_K - tL_b)(0)$ . Since  $b$  is a root of a monic polynomial, the operator  $L_b \in \text{End}_k(K)$  can be represented by a matrix with coefficients in  $A$ . Therefore, for any  $b \in B$  integral over  $A$ , the trace of  $b$  is integral over  $A$  **hence belongs to  $A$** .

**Step 2:** The bilinear symmetric form  $x, y \rightarrow \text{Tr}(xy)$  is non-degenerate. Indeed,  $\text{Tr}(xx^{-1}) = \dim_{k(A)} K$ , and  $\text{char } k(A) = 0$ .

## Finiteness of integral closure for Noetherian rings (2)

**Step 2:** The bilinear symmetric form  $x, y \rightarrow \text{Tr}(xy)$  is non-degenerate. Indeed,  $\text{Tr}(xx^{-1}) = \dim_{k(A)} K$ , and  $\text{char } k(A) = 0$ .

**Step 3:** Choose a basis  $e_1, \dots, e_n$  in the  $k(A)$ -vector space  $K$ . Let  $P_i(t) \in k(A)[t]$  be the minimal polynomials of  $e_i$ . Write  $P_i(t) = A_i t^{n_i} + \sum_{j < n_i} a_{ij} t^j$ , where  $A_i, a_{ij} \in A$ . Then  $A_i e_i$  is a root of a monic polynomial  $\tilde{P}_i(t) = t^{n_i} + \sum_{j < n_i} A_i^{-1} a_{ij} t^j$ . **This proves that the basis  $e_1, \dots, e_n$  in  $K : k(A)$  can be chosen such that all  $e_i$  are integral over  $A$ , that is, all  $e_i$  belong to  $B$ .**

**Step 4:** Let  $e_i^* \in K$  be the dual basis with respect to the form  $\text{Tr}$ , with  $\text{Tr}(e_i^* e_j) = \delta_{ij}$ . Consider the  $A$ -module  $M \subset K$  generated by  $e_i^*$ . Clearly,  $M := \{b \in K \mid \text{Tr}(be_i) \in A\}$ .

**Step 5:** For any  $b \in B$ , the trace  $\text{Tr}(be_i)$  belongs to  $A$ , because  $be_i$  is integral over  $A$  (Step 1). Then  $B \subset M$ , and  **$B$  is a submodule of a finitely generated  $A$ -module  $M$** . Since  $A$  is Noetherian,  $B$  is finitely generated as an  $A$ -module.

■

## The normalization

**REMARK:** The ring  $\mathbb{C}[z_1, \dots, z_n]$  is factorial by Gauss lemma, and therefore integrally closed.

**COROLLARY:** Let  $B$  be a ring over  $\mathbb{C}$ . Assume that there exists an injective ring morphism from  $A = \mathbb{C}[x_1, \dots, x_k]$  to  $B$  such that  $B$  is finitely generated as an  $A$ -module. Then its integral closure  $\tilde{B}$  is a finitely generated  $A$ -module. In particular,  $\tilde{B}$  is a finitely generated ring.

**Proof:** Since  $A$  is factorial, it is integrally closed, and Theorem 1 applies. ■

**DEFINITION:** Let  $X$  be an irreducible affine variety, and  $\tilde{A}$  the integral closure of its ring of regular functions. Assume that  $\tilde{A}$  is a finitely generated ring. Then  $\tilde{X} := \text{Spec}(\tilde{A})$  is called the normalization of  $X$ .

**EXAMPLE:** Let  $C \subset \mathbb{C}^2$  be a curve defined by equations  $x^2 = y^3$ , and  $t := x/y$ . Then  $t^3 = x^3/y^3 = y^3x/y^3 = x$ , hence  $t$  belongs to the integral closure of  $\mathcal{O}_C$ . Since  $x = t^2$ ,  $y = y^3/y^2 = x^2/y^2 = t^2$ , the ring  $\mathbb{C}[t]$  contains  $\mathcal{O}_C$ . Since this ring is factorial, it is integrally closed, hence it coincides with the integral closure of  $\mathcal{O}_C$ .

## Normalization (2)

**COROLLARY:** Let  $X$  be an affine variety, and  $\tilde{A}$  the integral closure of its ring of regular functions. **Then the ring  $\tilde{A}$  is finitely generated.**

**Proof. Step 1:** The variety  $X$  admits a finite, dominant map to  $\mathbb{C}^k$ . Let  $A$  be the integral closure of  $\mathbb{C}[z_1, \dots, z_n]$  in  $k(X)$ ; it is a finitely generated algebra by the previous theorem. Then  $A$  is an integrally closed ring containing  $\mathcal{O}_X$  and with the same field of fractions.

**Step 2: It remains to show that  $A = \tilde{A}$ .** Since  $A \supset \mathcal{O}_X \supset \mathbb{C}[z_1, \dots, z_n]$ , we obtain that  $A$  is a finitely generated module over  $\mathcal{O}_X$ . Therefore,  $A \subset \tilde{A}$ ; since  $A \supset \mathcal{O}_X$  is integrally closed,  $A$  contains the integral closure of  $\mathcal{O}_X \subset k(X)$ , which gives  $A \supset \tilde{A}$ . ■

**DEFINITION:** Let  $X$  be an affine variety, and  $\tilde{A}$  the integral closure of its ring of regular functions. Then  $\tilde{X} := \text{Spec}(\tilde{A})$  is called **normalization of  $X$** .

## Normalization (3)

**DEFINITION:** Let  $X$  be an affine variety, and  $\tilde{A}$  the integral closure of its ring of regular functions. Then  $\tilde{X} := \text{Spec}(\tilde{A})$  is called **normalization of  $X$** .

**REMARK:** The normalization map is finite and birational;  $X$  is normal if for any finite, birational  $\varphi : X' \rightarrow X$ , the map  $\varphi$  is an isomorphism. Indeed, in this case  $\mathcal{O}_{X'} \supset \mathcal{O}_X$  is finite with the same field of fractions.

**COROLLARY:** Normalization of  $X$  is a finite, birational morphism  $X' \rightarrow X$  such that for any other finite, birational  $\varphi : X'' \rightarrow X'$ , the map  $\varphi$  is an isomorphism. In particular, any birational, finite map  $X' \rightarrow X$  with  $X'$  normal is a normalization. ■

**REMARK:** In other words, normalization is an initial object in the category  $\mathcal{C}$ , such that  $\text{Ob}(\mathcal{C})$  is the set of the pairs  $(X' \xrightarrow{\mu'} X)$  where the morphism  $\mu$  is finite and birational, and morphisms of  $\mathcal{C}$  are maps  $\varphi : X' \rightarrow X''$  making the following diagram commutative

$$\begin{array}{ccc}
 X' & \xrightarrow{\varphi} & X'' \\
 \mu' \searrow & & \swarrow \mu'' \\
 & X & \\
 & 17 & 
 \end{array}$$

## Finite quotients

**REMARK:** Let  $R$  be a finitely generated ring and  $G$  a finite group acting by automorphisms on  $R$ , and  $R^G$  the ring of  $G$ -invariants. **Then  $R^G$  is finitely generated** (Lecture 5).

**CLAIM:** Let  $R$  be a Noetherian ring without zero divisors,  $G$  a finite group acting by automorphisms on  $R$ , and  $R^G$  the ring of  $G$ -invariants. **Then  $\varphi : \text{Spec } R \rightarrow \text{Spec } R^G$  is a finite, dominant morphism.**

**Proof. Step 1:** For any  $g \in G$ , consider the corresponding polynomial map  $P_g : R \rightarrow R$ , and let  $r \in R$ . The polynomial  $P(t) := \prod_{g \in G} (t - g(r))$  has  $G$ -invariant coefficients for any  $r \in R$ , hence  $P(t) \in R^G[t]$ . **This implies that  $R$  is integral closure of  $R^G$  in  $k(R)$ .** By Theorem 1,  $R$  is finite over  $R^G$ .

**Step 2:** The morphism  $\varphi$  is finite because each  $r \in R$  satisfies the equation  $P(r) = 0$ , where  $P(t) = \prod_{g \in G} (t - g(r))$ . It is dominant, because  $R^G \subset R$ . ■

**DEFINITION:** Let  $G$  be a finite group acting on an affine variety  $X$  by automorphisms. **The quotient space  $X/G$  is  $\text{Spec}(\mathcal{O}_X^G)$ .**

**EXAMPLE:**  $\mathbb{C}^2/\{\pm 1\} = \mathbb{C}[x^2, y^2, xy] = \mathbb{C}[t_1, t_2, t_3]/(t_1 t_2 = t_3^2)$ . Indeed,  $\mathbb{C}^2/\{\pm 1\} = \text{Spec } A$ , where  $A = \mathbb{C}[x, y]^{\{\pm 1\}}$ . Clearly,  **$A$  is the ring of even polynomials.**

**EXAMPLE:** Let  $G = \mathbb{Z}/n\mathbb{Z}$  act on  $\mathbb{C}$  by multiplication by a primitive root  $\sqrt[n]{1}$ . Then  $\mathbb{C}/G = \text{Spec}(\mathbb{C}[t]^G) = \text{Spec}(\mathbb{C}[t^n])$ , hence **the quotient space  $\mathbb{C}/G$  is isomorphic to  $\mathbb{C}$ .**

## Finite quotients (2)

**THEOREM:** Consider the natural morphism  $\text{Spec } R \xrightarrow{\varphi} \text{Spec } R^G$ . Then  $\varphi(x) = \varphi(y)$  if and only if  $x \in G \cdot y$ , that is, the **set of points in  $\text{Spec } R^G$  is identified with the space of  $G$ -orbits.**

**Proof. Step 1:** If two maximal ideals of  $R$  are  $G$ -conjugated, their intersections of  $R^G \subset R$  are equal. This gives  $\varphi(gx) = \varphi(x)$ : **each  $G$ -orbit is mapped to one point.** It remains to show that **the preimage of any point is exactly one  $G$ -orbit.**

**Step 2:** For any ideal  $\mathfrak{m} \subset R^G$ , one has  $(\mathfrak{m}R)^G = \mathfrak{m}$  (lecture 5). Then  $A^G = R^G/\mathfrak{m}$ , where  $A := R \otimes_{R^G} (R^G/\mathfrak{m}) = R/\mathfrak{m}R$ .

**Step 3:** Let  $\mathfrak{m}$  be the maximal ideal of  $y \in \text{Spec } R^G$ , and  $N$  the nilradical of  $A := R/\mathfrak{m}R$ . Since  $\varphi^{-1}(y) = \text{Spec}(A/N)$ , **points of  $\varphi^{-1}(y)$  are maximal ideals of the ring  $A/N$ .**

**Step 4:** A semisimple Artinian  $\mathbb{C}$ -algebra  $A/N$  is a direct sum of finite extensions of  $\mathbb{C}$ , which are all isomorphic to  $\mathbb{C}$ , giving  $A/N = \bigoplus \mathbb{C}$ . Since  $A^G = \mathbb{C}$  (Step 2), **the group  $G$  acts on the summands of  $A/N = \bigoplus \mathbb{C}$  transitively.** Therefore, **all points of  $\varphi^{-1}(y)$  belong to the same  $G$ -orbit. ■**