

# Commutative Algebra

## lecture 13: Krull dimension and algebraic dimension

Misha Verbitsky

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## Algebraic dimension

**DEFINITION:** Let  $A \supset \mathbb{C}[t_1, \dots, t_n]$  be a ring without zero divisors. Assume that the extension  $[k(A) : \mathbb{C}(t_1, \dots, t_n)]$  is finite. Then  $n$  is called **the transcendence degree** of  $A$ .

**REMARK:** Let  $M$  be an irreducible affine variety, and  $\pi : M \rightarrow \mathbb{C}^n$  the finite, dominant map constructed in Noether's normalization lemma. By construction,  $n$  is equal to the transcendence degree of  $[k(M) : \mathbb{C}]$ .

**DEFINITION:** Let  $M$  be an irreducible affine variety. **Algebraic dimension**  $a(M)$  is the transcendence degree of  $[k(M) : \mathbb{C}]$ .

## Divisors in $\mathbb{C}^n$

**DEFINITION:** Let  $X$  be an affine variety, and  $f \in \mathcal{O}_X$  a regular function which does not vanish on any of irreducible components of  $X$ . The zero set of  $f$  is called **a principal divisor** on  $X$ . Its irreducible components are called **divisors** on  $X$ .

**LEMMA:** Any divisor in  $\mathbb{C}^n$  is principal.

**Proof:** Let  $I = (f) \subset \mathbb{C}[t_1, \dots, t_n]$  be a principal ideal,  $f = \prod_i f_i^{\alpha_i}$  be the prime decomposition of  $f$ , and  $V_{(f)}$ ,  $V_{(f_i)}$  be the corresponding zero sets. The ideals  $(f_i)$  are prime, because  $\mathbb{C}[t_1, \dots, t_n]$  is factorial. Since  $V_I = \bigcup V_{(f_i)}$ , and all the ideals  $(f_i)$  are prime, the decomposition  $V_I = \bigcup V_{(f_i)}$  coincides with the irreducible decomposition of  $V_{(f)}$ . We obtain that **any irreducible component of  $V_{(f)}$  is principal.** ■

## Algebraic dimension of a divisor

**Proposition 1:** Let  $D$  be an irreducible divisor in  $\mathbb{C}^n$ . Then  $a(D) = n - 1$ .

**Proof:** Let  $D = V_{(f)}$ , where  $f$  is an irreducible polynomial (it exists by the previous lemma). Then  $f = 0$  gives an algebraic relation between the coordinate functions, hence the maximal number of algebraically independent coordinate functions on  $D$  is  $n - 1$ . Moreover, for each of the coordinates, say,  $t_n$ , such that  $f$  depends non-trivially on  $t_n$ , its image in  $k(D)$  is a root of a polynomial with coefficients in  $\mathbb{C}[t_1, \dots, t_{n-1}]$ . Therefore  $k(D)$  is a finite extension of  $\mathbb{C}(t_1, \dots, t_{n-1})$ .

■

## Krull dimension

**REMARK: Length of a chain**  $A_1 \subset A_2 \subset A_3 \subset \dots \subset A_n$  is  $n - 1$ , that is, the number of  $\subset$  signs.

**DEFINITION: Krull dimension** of a ring  $A$  is the maximal possible length of a chain of prime ideals  $0 \neq \mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \dots \subsetneq \mathfrak{p}_n \subsetneq A$

**DEFINITION: Krull dimension** of a variety  $X$  is the maximal possible length of a chain of non-empty, irreducible, distinct subvarieties  $X_1 \subsetneq \dots \subsetneq X_n$ .

Today we are going to prove the following theorem

**THEOREM:** For any affine variety, **its algebraic dimension is equal to its Krull dimension.**

## Wolfgang Krull (1899-1971)



*Wolfgang Krull (1899-1971),  
photo by Paul Halmos,  
Seventh Brazilian Mathematics Colloquium  
in Poços de Caldas, Brazil, July 1969,*

## Local rings and Nakayama lemma

**DEFINITION:** A ring  $A$  is called **local** if it has only one maximal ideal.

**DEFINITION:** Let  $\mathfrak{p} \subset A$  be a prime ideal, and  $S \subset A$  its complement. **Localization of  $A$  in  $\mathfrak{p}$**  is  $A[S^{-1}]$ .

**CLAIM: Localization  $A_{\mathfrak{p}}$  of  $A$  in  $\mathfrak{p}$  is local.**

**Proof:** Any  $x \in A \setminus \mathfrak{p}$  is invertible, hence  $\mathfrak{p}$  is a maximal ideal, containing all ideals in  $A$ . ■

**THEOREM: (Nakayama's lemma for local rings)**

Let  $A$  be a Noetherian local ring,  $\mathfrak{p}$  its maximal ideal, and  $M$  a finitely generated  $A$ -module. Then  $M \not\supseteq \mathfrak{p}M$ .

**Proof:** For any non-trivial ideal  $\mathfrak{a} \subset A$ , **Nakayama lemma claims that  $\mathfrak{a}M = M$  implies that  $(1 + a)M = 0$** , for some  $a \in \mathfrak{a}$ . For any  $a \in \mathfrak{p}$ ,  $1 + a$  is invertible, hence  $M = 0$ . ■

## Finite ring extensions and prime ideals: going down

**DEFINITION:** Let  $B \supset A$  be a ring, which is finitely generated as an  $A$ -module. In this case, we say that  $B$  is **finite extension of  $A$** .

**Lemma 1:** Let  $B \supset A$  be an extension of a ring  $A$  without zero divisors, and  $\mathfrak{q} \subset B$  a non-zero prime ideal. **Then  $\mathfrak{p} := \mathfrak{q} \cap A$  is nonzero.**

**Proof:** Consider the ring  $A_{\mathfrak{p}} = A[S^{-1}]$  localized in the set  $S$  of all  $s \notin \mathfrak{p}$ , and let  $B_{\mathfrak{p}} := B[S^{-1}]$ . Then  $B_{\mathfrak{p}} \supset A_{\mathfrak{p}}$  is a finite  $A_{\mathfrak{p}}$ -module. If  $\mathfrak{p} = 0$ ,  $A_{\mathfrak{p}}$  is a field, and then  $B_{\mathfrak{p}}$  is also a field as follows from the classification of semisimple Artinian algebras over a field. However,  $B_{\mathfrak{p}}$  contains a non-trivial ideal  $\mathfrak{q} \neq 0$ , hence it cannot be a field. ■

## Finite ring extensions and prime ideals: going up

**Lemma 2:** Let  $B \supset A$  be a finite extension of a Noetherian ring  $A$ , and  $\mathfrak{p} \subset A$  a prime ideal. **Then there exists finitely many prime ideals  $\mathfrak{q} \subset B$  such that  $\mathfrak{p} = \mathfrak{q} \cap A$ .**

**Proof. Step 1:** As above, consider the ring  $A_{\mathfrak{p}} = A[S^{-1}]$  localized in the set  $S$  all  $s \notin \mathfrak{p}$ . The kernel of the natural map  $A \rightarrow A_{\mathfrak{p}}/\mathfrak{p}$  is  $\mathfrak{p}$ . Indeed, the map  $A/\mathfrak{p} \rightarrow A_{\mathfrak{p}}/\mathfrak{p}$  has no kernel because  $\mathfrak{p}$  is prime, and the kernel of  $A \rightarrow A/\mathfrak{p}$  is  $\mathfrak{p}$ . Therefore, **for any prime ideal  $\mathfrak{p}' \subset A$  such that  $A_{\mathfrak{p}}\mathfrak{p}' = A_{\mathfrak{p}}\mathfrak{p}$ , we have  $\mathfrak{p}' = \mathfrak{p}$ .**

**Step 2:** Let  $B_{\mathfrak{p}} := B[S^{-1}]$ . By Nakayama's lemma,  $B_{\mathfrak{p}} \neq \mathfrak{p}B_{\mathfrak{p}}$ . Then  $B_{\mathfrak{p}}/\mathfrak{p} = B_{\mathfrak{p}} \otimes_A A_{\mathfrak{p}}/\mathfrak{p}$  is a non-zero, finite-dimensional ring over the field  $A_{\mathfrak{p}}/\mathfrak{p}$ . Let  $\tilde{\mathfrak{q}} \subset B_{\mathfrak{p}}$  be the preimage of any prime ideal in  $B_{\mathfrak{p}}/\mathfrak{p}$  (there are finitely many prime ideals by the classification of Artinian algebras), and  $\mathfrak{q} := \tilde{\mathfrak{q}} \cap B$ . By construction,  $\tilde{\mathfrak{q}}$  is prime, contains  $\mathfrak{p}A_{\mathfrak{p}}$ , and  $\mathfrak{q}/\mathfrak{p}$  is a prime ideal in the Artinian algebra  $B_{\mathfrak{p}}/\mathfrak{p}$ . Therefore,  $A_{\mathfrak{p}} \cap \tilde{\mathfrak{q}} = A_{\mathfrak{p}}\mathfrak{p}$ . By Step 1, this implies  $A \cap \mathfrak{q} = \mathfrak{p}$ . ■

## Cohen-Seidenberg theorems

### THEOREM: (Cohen-Seidenberg theorem)

Let  $B \supset A$  be a finite Noetherian ring over  $A$ , and  $\mathfrak{q}_1 \subsetneq \mathfrak{q}_2 \dots \subsetneq \mathfrak{q}_n \subsetneq B$  be a chain of prime ideals. Denote by  $\mathfrak{p}_i$  the ideal  $\mathfrak{p}_i \cap A \subset A$ ; it is clearly prime. Then

**(i)  $\mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \dots \subsetneq \mathfrak{p}_n \subsetneq A$  (distinct prime ideals remain distinct)**

**(ii) Any chain of prime ideals  $\mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \dots \subsetneq \mathfrak{p}_n \subsetneq A$  is obtained this way.**

**Proof of (i):** Suppose that  $\mathfrak{p}_i = \mathfrak{p}_{i-1}$ . Replacing  $A$  by  $A/\mathfrak{p}_{i-1}$  and  $B$  by  $B/\mathfrak{q}_{i-1}$ , we reduce the statement of (i) to Lemma 1.

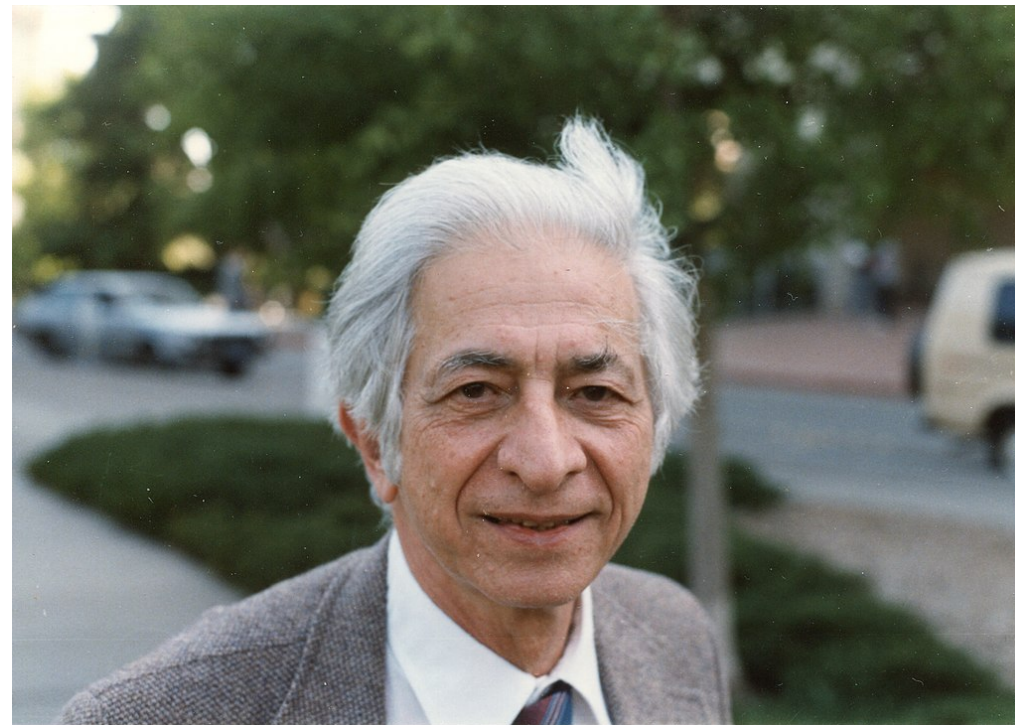
**Proof of (ii):** Existence of  $\mathfrak{q}_1$  follows from Lemma 2. Using induction, we may assume that  $\mathfrak{q}_1 \subsetneq \mathfrak{q}_2 \subsetneq \dots \subsetneq \mathfrak{q}_r$  is already chosen. To prove the induction step, we need to choose a prime ideal  $\mathfrak{q}_{r+1}$  in  $B/\mathfrak{q}_r$  such that  $\mathfrak{q}_{r+1} \cap A/\mathfrak{q}_r = \mathfrak{p}_{r+1}$ . This is again Lemma 2. ■

## Irving S. Cohen and Abraham Seidenberg

*Irving S. Cohen and Abraham Seidenberg, "Prime ideals and integral dependence", 1946, Bull. Amer. Math. Soc. 52 (4): 252-261*



Irvin Sol Cohen  
(1917-1955)



Abraham Seidenberg  
(1916-1988)

## Krull dimension is invariant under finite morphisms

**COROLLARY:** Let  $X \rightarrow Y$  be a finite, dominant morphism of irreducible affine varieties. **Then the Krull dimension of  $X$  is equal to Krull dimension of  $Y$ .**

**Proof:** Any chain of prime ideals in  $\mathcal{O}_Y \subset \mathcal{O}_X$  can be lifted to  $\mathcal{O}_X$  by Cohen-Seidenberg; any chain of distinct prime ideals in  $\mathcal{O}_X$  intersected with  $\mathcal{O}_Y$  gives a chain of distinct prime ideals in  $\mathcal{O}_Y$ , again by Cohen-Seidenberg. ■

## The Krull dimension and the algebraic dimension

**THEOREM:** For any affine variety  $X$ , **its algebraic dimension  $a(X)$  is equal to its Krull dimension  $\dim X$ .**

**Proof. Step 1:** By Noether's lemma, there exists a finite, dominant map  $X \rightarrow \mathbb{C}^d$ , where  $d = a(X)$  is the algebraic dimension. It remains to show that the Krull dimension of  $\mathbb{C}^d$  is  $d$ .

**Step 2:** Any prime ideal which is not principal contains an irreducible polynomial, hence it contains a principal prime ideal. This implies that **in any maximal chain  $0 \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \dots$  of prime ideals in  $\mathcal{O}_{\mathbb{C}^n}$ , the ideal  $\mathfrak{p}_1$  is principal.**

**Step 3:** We obtained that  $\dim \mathbb{C}^n = 1 + \dim D$ , where  $D$  is an irreducible divisor. Using induction in  $\dim X$ , we may assume that  $\dim Y = a(Y)$  for any affine variety of algebraic dimension  $< n$ . Then  $\dim D = a(D) = n - 1$  (Proposition 1), giving  $\dim \mathbb{C}^n = \dim D + 1 = n$ . ■

## The proof of Hilbert Nullstellensatz

Using Noether's normalization lemma, we can prove the result which can be used to deduce Hilbert Nullstellensatz. In Lecture 1, we proved this theorem using the set theory.

**THEOREM:** Let  $F$  be a finitely generated algebra over an algebraically closed field  $\bar{k}$ . **Assume that  $F$  is a field. Then  $F = \bar{k}$ .**

**Proof:** Noether normalization lemma implies that there exists an embedding  $\bar{k}[t_1, t_2, \dots, t_n] \hookrightarrow F$  such that  $F$  is finite generated as  $\bar{k}[t_1, t_2, \dots, t_n]$ -module. By Lemma 1, this implies that **every non-zero prime ideal in  $\bar{k}[t_1, t_2, \dots, t_n]$  is extended to a non-zero prime ideal in  $F$** . However,  $F$  is a field, hence it has no non-zero ideals, and  $n = 0$ . Therefore,  $F$  is a finite extension of  $\bar{k}$ . Since  $\bar{k}$  is algebraically closed, any finite extension of  $\bar{k}$  is trivial. ■