Commutative Algebra

lecture 14: Fractional ideals

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http://verbit.ru/IMPA/CA-2026/

IMPA, sala 236

May 18, 2026

Fractional ideals: basic properties

DEFINITION: Let R be a ring without zero divisors, and $k(R)$ its fraction field. A non-zero R-submodule $I \subset k(R)$ is called **a fractional ideal** of R if for some $a \in R$, one has $aI \subset R$.

CLAIM: Let R be a Noetherian ring, and $I \subset k(R)$ an R-submodule. Then **I is a fractional ideal if and only if I is finitely generated.**

Proof: Let I be finitely generated by the collection $\{\frac{a_i}{b_i} \in k(R)\}$, with $a_i, b_i \in R$. Then $\prod_i b_i I \subset R$. Conversely, if $aI \subset R$, then aI is finitely generated, because aI is an ideal in a Noetherian ring. ■

DEFINITION: Let I_1, I_2 be fractional ideals. Then the **set $I_1 I_2$** of products of elemens in I_1, I_2 is a fractional ideal.

CLAIM: For any two fractional ideals I_1, I_2, **the intersection $I_1 \cap I_2$ is non-empty,** hence **$I_1 \cap I_2$ is also a fractional ideal.**

Proof: Since $aI_i \subset R$, the intersection $I_i \cap R$ is non-empty. Let $a_i \in I_i \cap R$. Then $a_1 a_2 \in RI_i = I_i$, hence $a_1 a_2 \in I_1 \cap I_2$. ■

Fractional ideals: I^{-1} and $R(I)$

CLAIM: Let $I \subset R$ be a fractional ideal. Then the sets $I^{-1} := \{x \in R \mid xI \subset R\}$ and $R(I) := \{x \in k(R) \mid xI \subset I\}$ are fractional ideals. Moreover, for any fractional ideal I_1, I_2, **the R-module $J := \{x \in k(R) \mid xI_1 \subset I_2\}$ is a fractional ideal.**

Proof. Step 1: Let $a, b \in R \backslash 0$ be elements such that $aI_1 \subset R$ and $b \in I_2 \cap R$. Then $abI_1 \subset bR \subset I_2$, hence J is non-empty.

Step 2: For any non-zero elements $c, d \in R$ such that $c \in I_1$, $dI_2 \subset R$, and any $x \in J$, we have $cdx = d(xc) \subset dI_2 \subset R$, hence $cdx \in R$. ■

Claim 1: For any fractional ideal I, **one has $R(I) \supset R \supset II^{-1}$. If, in addition, $I \subset R$, then $I^{-1} \supset R$.**

Proof: $R(I) \supset R$ because I is R-module and $R \supset II^{-1}$ because $aI \subset R$ for any $a \in I^{-1}$. Finally, $I \subset R$ implies that any $x \in R$ satisfies $xI \subset R$, hence $x \in I^{-1}$.

■

REMARK: $I^{-1} = \operatorname{Hom}_R(I, R)$ and $R(I) = \operatorname{Hom}_R(I, I)$. Indeed, any R-linear map $\varphi \in \operatorname{Hom}(I_1, I_2)$, if tensored with $k(R)$, becomes an element of $\operatorname{Hom}(k(R), k(R)) = k(R)$. Therefore, **$\varphi$ is a multiplication by $z \in k(R)$ which satisfies $zI_1 = I_2$.**

REMARK: If R is Noetherian and integrally closed, $R(I) = R$. Indeed, $R(I) = \operatorname{Hom}_R(I, I)$ is a finitely generated R-module in $k(R)$.

Projective R-modules

DEFINITION: An R-module M is called **free** if M is a direct sum of several copies of R (possibly infinitely many copies). It is called **projective** if it is a direct summand of a free R-module.

PROPOSITION: An R-module P is projective if for every surjective homomorphism $\varphi: \ A \longrightarrow B$ of R-modules and every homomorphism $\psi: \ P \longrightarrow B$, **the map ψ can be factorized through φ** making the following diagram commutative:

$$\begin{tikzcd}
 & A \arrow[d, "\varphi"] \\
P \arrow[ur, dashed, "\mu"] \arrow[r, "\psi"] & B
\end{tikzcd}$$

Proof: Let $\varphi: \ F \longrightarrow P$ be a surjective map from a free module to P and $\psi = \varphi$. The map ψ **can be factorized through φ if and only if ψ admits a section μ,** which gives a decomposition $F = \ker \psi \oplus \operatorname{im} \mu$.

Conversely, if P is a direct summand of $F = P \oplus P_1$, we can extend ψ from P to a free R-module $F = P \oplus P_1$. Then **the map μ can be defined on the generators of F and restricted to $P \subset F$.** ■

Dual basis theorem

THEOREM: (Dual basis theorem) Let M be an R-module. Consider a natural map $\Psi: \operatorname{Hom}_R(M,R)\otimes_R M \longrightarrow \operatorname{Hom}_R(M,M)$. **Then the following are equivalent.**

(i) Ψ is an isomorphism.

(ii) $\operatorname{Id}_M \in \operatorname{im}(\Psi)$.

(iii) M is projective and finitely generated.

Proof. Step 1: Clearly, Ψ is an isomorphism for any free finitely generated R-module M, hence for any direct sum component of a free finitely generated R-module. **Therefore, (iii) $\Rightarrow$ (i) $\Rightarrow$ (ii).**

Step 2: The condition (ii) is equivalent to the following. There exists a finite collection of maps $f_i: M \longrightarrow R$, $i = 1, ..., n$ and a finite set $m_i \in M$, $i = 1, ..., n$ such that for any $m \in M$ one has $\sum_{i=1}^n f_i(m)m_i = m$. In particular, **(ii) implies that M is finitely generated.**

Step 3: Let $F = \langle m_i \rangle$ be a free module generated by $\{m_i\}$, and $f: M \to F$ the map $f(m) \mapsto \sum_{i=1}^n f_i(m)m_i$. Then f is a section of the natural surjective map $F \longrightarrow M$, hence M is projective. **This gives (ii) $\Rightarrow$ (iii).** ■

Projective modules in local rings

PROPOSITION: Let P be a finitely generated projective module over a local ring R. **Then P is a free R-module.**

Proof: Denote by $\mathfrak{m}$ the maximal ideal of R. Let $e_1, ..., e_n$ be a set of elements of $P\backslash\mathfrak{m}P$ generating $\frac{P}{\mathfrak{m}P}$. We chose e_i in such a way that their images in $\frac{P}{mP}$ are linearly independent over the field $R/\mathfrak{m}$. By Nakayama's lemma, $\{e_i\}$ generate P. Since P is projective, the projection from the free module $F := \langle e_1, ..., e_m\rangle \longrightarrow P$ has a section $\psi : P \longrightarrow \langle e_1, ..., e_m\rangle$. **Since $\operatorname{im}\psi$ modulo $\mathfrak{m}F$ generates $\frac{F}{\mathfrak{m}F}$, we have $F = \operatorname{im}\psi$, and P is a free R-module.** ■

Invertible fractional ideals

DEFINITION: A fractional ideal $I \subset k(R)$ is called **invertible** if $II^{-1} = R$.

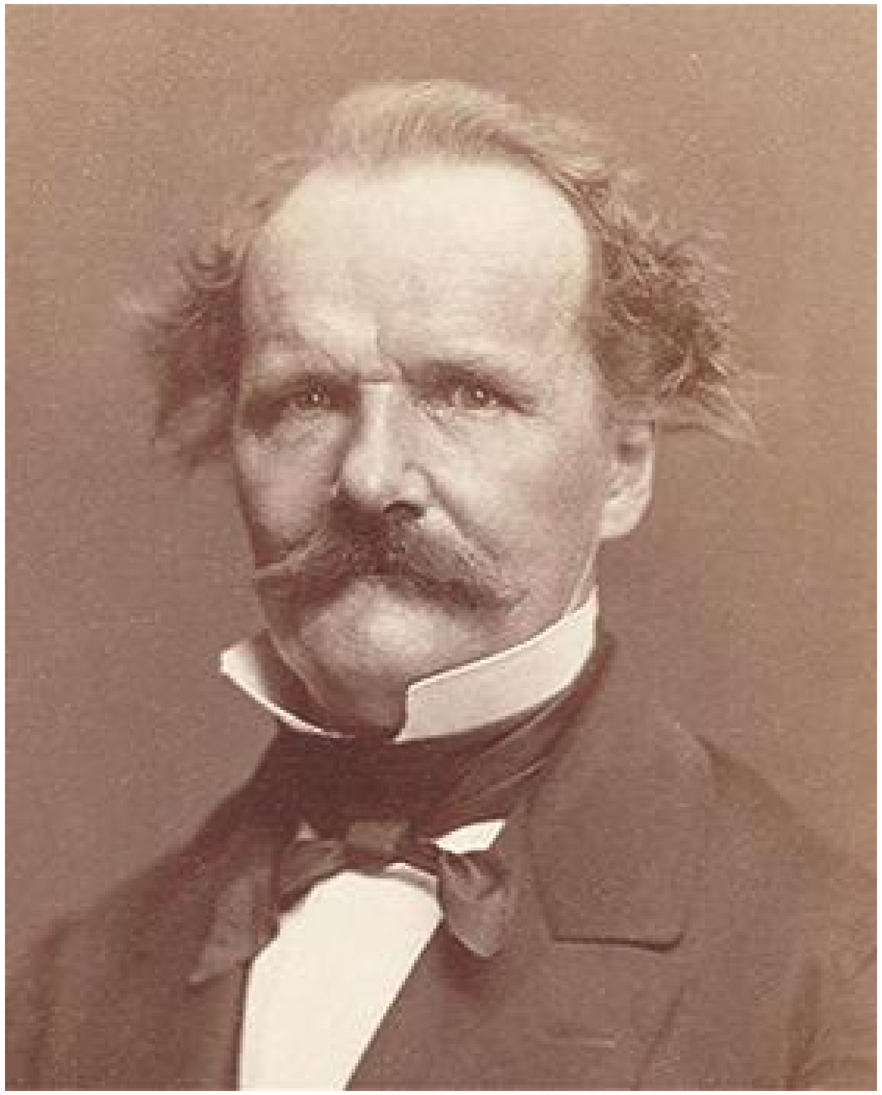
THEOREM: A fractional ideal is invertible if and only if it is projective, and it is then finitely generated.

Proof. Step 1: Clearly, $I^{-1} \subset \operatorname{Hom}_R(I,R)$. Therefore, $I^{-1}I \ni 1$ implies $\operatorname{Id}_I \in \operatorname{Hom}_R(I,R) \otimes_R I$, which implies projectivity by the Dual Basis Theorem.

Step 2: Consider an R-linear map $\varphi : I_1 \longrightarrow I_2$. If we tensor φ it with $k(R)$, we obtain a $k(R)$-linear map $I_1 \otimes_R k(R) \longrightarrow I_2 \otimes_R k(R)$, with $I_i \otimes_R k(R) = k(R)$. Clearly, $\operatorname{Hom}_{k(R)}(k(R), k(R)) = k(R)$. **Therefore, φ it is expressed as $v \longrightarrow \alpha v$, for some $\alpha \in k(R)$.** This implies that the natural map $I^{-1} \to \operatorname{Hom}_R(I,R)$ is an isomorphism.

Step 3: If I is projective, then $\operatorname{Id}_I \in \operatorname{Hom}_R(I,R) \otimes_R I$, hence $I^{-1}I \ni 1$, and $I^{-1}I = R$. ■

Ernst Kummer (1810 - 1893)



Ernst Eduard Kummer (1810-1893), a German algebraist, was rather poor at arithmetic. Once he had to find 7 x 9. "Seven times nine," he began, "Seven times nine is er – ah — ah – seven times nine is. . . ." "Hmmm the product cannot be 61, because 61 is prime, it cannot be 65, because 65 is a multiple of 5, 67 is a prime, 69 is too big - Only 63 is left."

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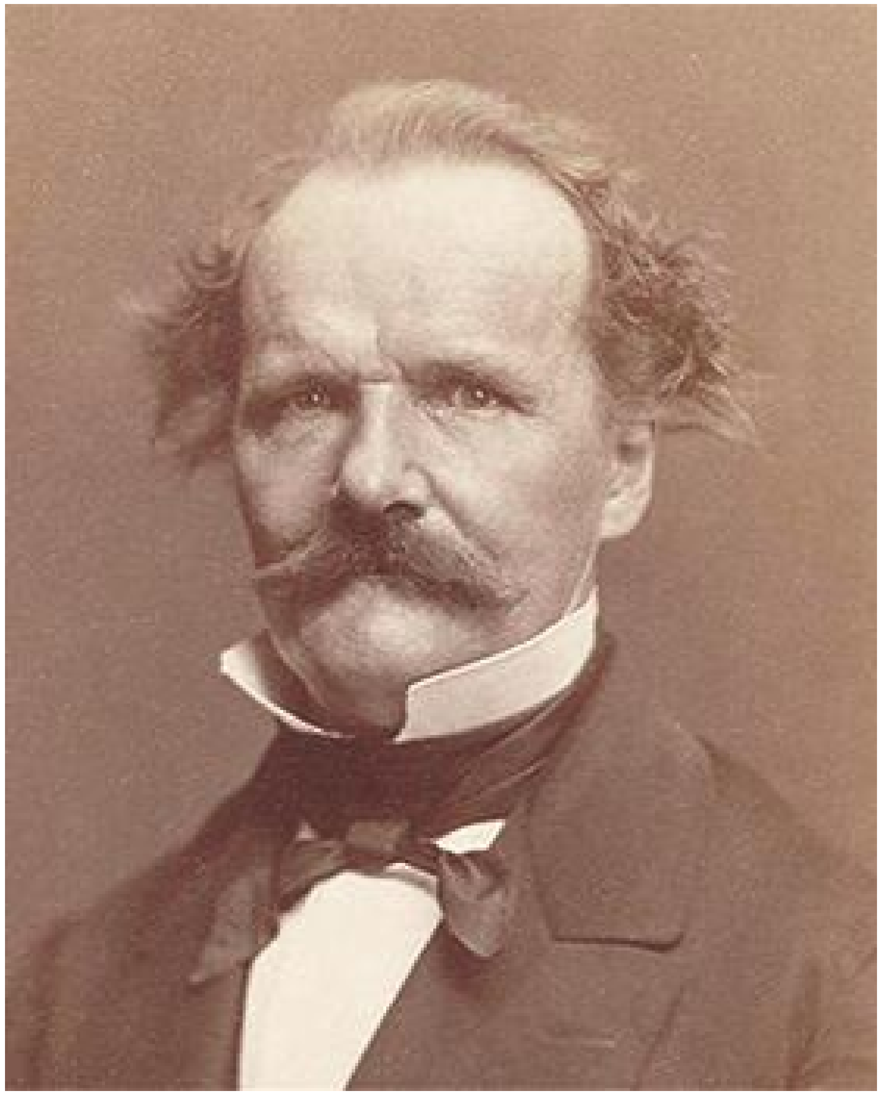

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Richard Dedekind (1831 - 1916)

Richard Dedekind's bas-relief at the main entrance of TU Braunschweig, Germany

Group structure on invertible fractional ideals

THEOREM: The multiplication of fractional ideals is associative, and this multiplication **induces the structure of an abelian group on the set of invertible fractional ideals in R.**

Proof: Let $I, J, K \subset k(R)$ be fractional ideals. Clearly, $I(JK) = (IJ)K$, and $IR = RI = I$, hence R is the unity of the semigroup of fractional ideals. Finally, $II^{-1} = R$ whenever I is an invertible ideal. ■