

# Commutative Algebra

## lecture 15: Discrete valuation rings

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## Discrete valuation rings

**DEFINITION:** Let  $K$  be a field. A **discrete valuation** on  $K$  is a surjective map  $\nu : (K \setminus 0) \rightarrow \mathbb{Z}$  such that  $\nu(xy) = \nu(x) + \nu(y)$  and  $\nu(x + y) \geq \min(\nu(x), \nu(y))$ . The **valuation ring** of  $\nu$  is  $\{x \in K \mid \nu(x) \geq 0\}$ .

**EXAMPLE:** Let  $R$  be a factorial ring, and  $p \in R$  a prime. Given  $x \in k(R)$ , write the prime decomposition  $x = \prod_i p_i^{\alpha_i}$  (here  $\alpha_i \in \mathbb{Z}$  and **can be negative**, because  $x$  belongs to the field of fractions). Take  $\nu_{p_1}(x) := \alpha_1$ . This function is called  **$p$ -adic valuation**. The corresponding valuation ring is all fractions  $\frac{a}{b}$  where  $a$  and  $b \in R$  are coprime and  $b$  is not divisible by  $p$ .

**DEFINITION:** A ring  $R$  without zero divisors is a **discrete valuation ring (DVR)** if  $k(R)$  admits a discrete valuation such that  $R$  is its valuation ring.

**EXAMPLE:** The ring  $k[[t]]$  of formal power series is a DVR.

**EXAMPLE:** The ring  $\mathbb{Z}_p$  of  $p$ -adic integers is a DVR.

## Discrete valuation rings and principal ideals

**Claim 1:** Let  $R$  be a Noetherian local ring without zero divisors, and  $\mathfrak{m}$  its maximal ideal. **Then  $\mathfrak{m}$  is principal if and only if  $R$  is a discrete valuation ring.**

**Proof:** Let  $R$  be a DVR. Consider an element  $p$  with  $\nu(p) = 1$ . For any  $x \in R$  with  $\nu(x) = k$ , one has  $x = up^k$ , where  $u$  is invertible, hence all ideals of  $R$  are principal.

Conversely, if  $\mathfrak{m}$  is generated by  $p$ , **any element  $x \in R$  which is not divisible by  $p$  is invertible**, hence  $x = up^k$ , where  $u$  is invertible. Then  $R$  is the DVR for the  $p$ -adic valuation. ■

**Corollary 1:** **A discrete valuation ring is local, Noetherian, integrally closed, all its ideals are principal, and the Krull dimension of  $R$  is 1.** ■

## Many ways to characterise a discrete valuation ring

**THEOREM:** Let  $R$  be a Noetherian local ring without zero divisors,  $\mathfrak{m}$  its maximal ideal, and  $k := R/\mathfrak{m}$  its residue field. Assume that  $R$  has Krull dimension 1. **Then the following are equivalent.**

- (i)  $R$  is a discrete valuation ring.
- (ii)  $R$  is integrally closed.
- (iii)  $\mathfrak{m}$  is a principal ideal.
- (iv)  $\dim_k \frac{\mathfrak{m}}{\mathfrak{m}^2} = 1$ .
- (v) Every non-zero ideal of  $R$  is a power of  $\mathfrak{m}$ .
- (vi) There exists  $p \in R$  such that every non-zero ideal of  $R$  is generated by  $p^k$ , for some  $k \in \mathbb{Z}^{>0}$ .

**Proof:** Later today.

## Fractional ideals (reminder)

**DEFINITION:** Let  $R$  be a ring without zero divisors, and  $k(R)$  its fraction field. A non-zero  $R$ -submodule  $I \subset k(R)$  is called **a fractional ideal** of  $R$  if for some  $a \in R$ , one has  $aI \subset R$ .

**CLAIM:** Let  $R$  be a Noetherian ring, and  $I \subset k(R)$  an  $R$ -submodule. Then  **$I$  is a fractional ideal if and only if  $I$  is finitely generated.**

**DEFINITION:** Let  $I_1, I_2$  be fractional ideals. Then the **set  $I_1 I_2$**  of products of elements in  $I_1, I_2$  is a fractional ideal.

**CLAIM:** For any two fractional ideals  $I_1, I_2$ , **the intersection  $I_1 \cap I_2$  is non-empty**, hence  **$I_1 \cap I_2$  is also a fractional ideal.**

**CLAIM:** Let  $I \subset R$  be a fractional ideal. Then the sets  $I^{-1} := \{x \in R \mid xI \subset R\}$  and  $R(I) := \{x \in k(R) \mid xI \subset R\}$  are fractional ideals. Moreover, for any fractional ideal  $I_1, I_2$ , **the  $R$ -module  $J := \{x \in k(R) \mid xI_1 \subset I_2\}$  is a fractional ideal.**

**CLAIM:** For any fractional ideal  $I$ , **one has  $R(I) \supset R \supset II^{-1}$ . If, in addition,  $I \subset R$ , then  $I^{-1} \supset R$ .**

## Invertible fractional ideals (reminder)

**DEFINITION:** A fractional ideal  $I \subset k(R)$  is called **invertible** if  $II^{-1} = R$ .

**THEOREM:** A fractional ideal is invertible if and only if it is projective, and it is then finitely generated.

**Proof. Step 1:**  $I^{-1}I \ni 1$  implies  $\text{Id}_I \in \text{Hom}_R(I, R) \otimes_R I$ , which implies projectivity by the Dual Basis Theorem.

**Step 2:** Consider an  $R$ -linear map  $\varphi : I_1 \rightarrow I_2$ . If we tensor  $\varphi$  it with  $k(R)$ , we obtain a  $k(R)$ -linear map  $I_1 \otimes_R k(R) \rightarrow I_2 \otimes_R k(R)$ , with  $I_i \otimes_R k(R) = k(R)$ . Clearly,  $\text{Hom}_{k(R)}(k(R), k(R)) = k(R)$ . **Therefore,  $\varphi$  is expressed as  $v \rightarrow \alpha v$ , for some  $\alpha \in k(R)$ .** This gives  $I^{-1} = \text{Hom}_R(I, R)$ .

**Step 3:** If  $I$  is projective, then  $\text{Id}_I \in \text{Hom}_R(I, R) \otimes_R I$ , hence  $I^{-1}I \ni 1$ , and  $I^{-1}I = R$ . ■

**REMARK:** The multiplication of fractional ideals is associative, and this multiplication **induces the structure of an abelian group on the set of invertible fractional ideals in  $R$ .**

## Radical ideals in local rings of Krull dimension 1

Recall that **the radical**  $\sqrt{I}$  of an ideal  $I \subset R$  is the set of all  $x \in R$  such that  $x^n \in I$  for some  $n > 0$ . An ideal  $I$  is called **radical** if  $I = \sqrt{I}$ .

**CLAIM:** Let  $R$  be a local ring of Krull dimension 1. **Then any non-zero radical ideal of  $R$  is its maximal ideal.**

**Proof:** Let  $I \subset R$  be a non-zero radical ideal. Then  $0$  in  $R/I$  is an intersection of all prime ideals in  $R/I$ . In other words,  $I$  is an intersection of prime ideals containing  $I$ . Since  $R$  is a ring of Krull dimension 1, any chain of prime ideals has length  $\leq 2$ . Since  $I$  is non-zero, there exists only one prime ideal  $\mathfrak{m} \subset R$  containing  $I$ , hence  $I = \mathfrak{m}$ . ■

**Corollary 2:** Let  $R$  be a Noetherian local ring of Krull dimension 1, and  $\mathfrak{m}$  its maximal ideal. **Then for any  $t \in \mathfrak{m}$  there exists  $n$  such that  $\mathfrak{m}^n \subset (t)$ .**

**Proof:** The radical of  $(t)$  is  $\mathfrak{m}$ , by the previous claim. ■

## Local rings of Krull dimension 1 with projective maximal ideal

**PROPOSITION:** Let  $R$  be a Noetherian local ring, and  $\mathfrak{m}$  its maximal ideal. **Assume that  $\mathfrak{m}$  is projective as an  $R$ -module. Then  $R$  is DVR.**

**Proof:** Since  $R$  is Noetherian and local,  $\mathfrak{m}$  is free. Choose the elements  $p_1, p_2, \dots, p_n$  freely generating  $\mathfrak{m}$ . Then  $\mathfrak{m} = \bigoplus_i (p_i)$ , which is impossible, unless  $n = 1$ , because  $p_1 p_2 \in (p_1) \cap (p_2)$ . **Therefore,  $\mathfrak{m}$  is a principal ideal,** and  $R$  is DVR by Claim 1. ■

**Corollary 3:** Let  $R$  be a Noetherian local ring, and  $\mathfrak{m}$  its maximal ideal. **Assume that  $\mathfrak{m}$  is invertible as a fractional ideal. Then  $R$  is DVR.**

**Proof:** Invertible fractional ideals are projective. ■

## Integrally closed local rings of Krull dimension 1

**Proposition 2:** Let  $R$  be an integrally closed Noetherian local ring of Krull dimension 1. **Then  $R$  is a discrete valuation ring.**

**REMARK:** The converse statement is implied by Corollary 1.

**Proof. Step 1:** Let  $I \subset k(R)$  be a fractional ideal, and  $R(I) \subset k(R)$  the ring of all  $x \in k(R)$  such that  $xI \subset I$ . **Since  $R(I) \subset \text{Hom}_R(I, I)$ , it is finitely generated as  $R$ -module, hence  $R(I) = R$ .**

**Step 2:** Let  $\mathfrak{m}$  be the maximal ideal of  $R$ . Then  $R \subset \mathfrak{m}\mathfrak{m}^{-1} \subset \mathfrak{m}$ . Since any element of  $R \setminus \mathfrak{m}$  is invertible, we have either  $\mathfrak{m}\mathfrak{m}^{-1} = R$  or  $\mathfrak{m}\mathfrak{m}^{-1} = \mathfrak{m}$ . **In the second case,  $\mathfrak{m}^{-1} = R$  by Step 1, because in this case  $\mathfrak{m}^{-1} \subset R(\mathfrak{m})$ . In the first case, the fractional ideal  $\mathfrak{m}$  is invertible.**

**Step 3:** By Krull lemma,  $\mathfrak{m} \supsetneq \mathfrak{m}^2 \supsetneq \dots \supsetneq \mathfrak{m}^n$ . Let  $t \in \mathfrak{m} \setminus \mathfrak{m}^2$ . By Corollary 2,  $\mathfrak{m}^n \subset (t)$ , for some  $n > 0$ ; choose the minimal  $n$  with this property, and let  $x \in \mathfrak{m}^{n-1} \setminus (t)$ .

Let  $y := \frac{x}{t}$ . Since  $x \notin (t)$ , we have  $y \notin R$ .

Clearly,  $x\mathfrak{m} \subset \mathfrak{m}^n \subset (t)$ . This gives  $y\mathfrak{m} \subset R$ , hence  $\mathfrak{m}^{-1} \neq R$ . Then Step 2 implies that  $\mathfrak{m}^{-1}\mathfrak{m} = R$ , and Corollary 3 implies that  $R$  is DVR. ■

## Residue fields of discrete valuation rings

**Claim 2:** Let  $R$  be a discrete valuation ring, and  $\mathfrak{m}$  its maximal ideal, and  $k := R/\mathfrak{m}$  its residue field. **Then  $\mathfrak{m}^d/\mathfrak{m}^{d+1}$  is 1-dimensional as a  $k$ -vector space for all  $d \in \mathbb{Z}^{\geq 0}$ .**

**Proof:** Let  $p$  be the generator of  $\mathfrak{m}$ , and  $L_{p^d}(x) := p^d x$ . Then  $L_{p^d} : R \rightarrow \mathfrak{m}^d$  is an isomorphism which maps  $\mathfrak{m}$  to  $\mathfrak{m}^{d+1}$ . ■

**REMARK:** For some discrete valuation rings, **the map  $R \rightarrow R/\mathfrak{m} = k$  has a section  $k \rightarrow R$** . This is true, for example, for the ring  $\mathbb{C}[t]$  localized in  $(t)$ ; in this case,  $k = \mathbb{C}$ . For other rings, such section does not exist; for example, consider the ring  $\mathbb{Z}$  localized in  $(p)$ ; in this case, the residue field is  $\mathbb{Z}/p\mathbb{Z}$  **(prove this)**.

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- (i)  $R$  is a discrete valuation ring.
- (ii)  $R$  is integrally closed.
- (iii)  $\mathfrak{m}$  is a principal ideal.
- (iv)  $\dim_k \frac{\mathfrak{m}}{\mathfrak{m}^2} = 1$ .
- (v) Every non-zero ideal of  $R$  is a power of  $\mathfrak{m}$ .
- (vi) There exists  $p \in R$  such that every non-zero ideal of  $R$  is generated by  $p^k$ , for some  $k \in \mathbb{Z}^{>0}$ .

**Proof. Step 1:** (i) implies (ii), (iii), (v) and (vi) as follows from Proposition 1. (i) implies (iv) by Claim 2. **Therefore, (i) implies (ii)-(vi).**

**Step 2:** (iii)  $\Leftrightarrow$  (i) by Claim 1. (ii) implies (i) by Proposition 2. Also, (v) implies (vi) which implies that  $R$  is factorial which implies (ii). **We proved that all (i)-(vi) are equivalent, except maybe (iv).**

**Step 3:** Now we prove that (iv) implies (iii). Let  $p$  be a generator of  $\mathfrak{m}$  modulo  $\mathfrak{m}^2$ . Then the natural map  $\frac{(p)}{(p)\mathfrak{m}} \rightarrow \frac{\mathfrak{m}}{\mathfrak{m}^2}$  is surjective. By Nakayama lemma, for any morphism  $\varphi : M \rightarrow N$  of finitely generated modules over a local ring, **surjectivity of the map  $\varphi : \frac{M}{\mathfrak{m}M} \rightarrow \frac{N}{\mathfrak{m}N}$  implies the surjectivity of  $\varphi$** , hence  $\mathfrak{m}$  is a principal ideal. ■