Commutative Algebra

lecture 16: Dedekind rings

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Dedekind rings

DEFINITION: Let R be a ring without zero divisors. Assume that all ideals $I \subset R$ are invertible as fractional ideals. Then R is called **a Dedekind ring**, of **a Dedekind domain**.

EXAMPLE: A principal ideal ring R without zero divisors is clearly Dedekind. Indeed, all its ideals are free R-modules, hence projective, hence invertible (Lecture 14).

LEMMA: Let $\mathfrak{p} \subset R$ be an ideal over a ring. Assume that $\mathfrak{p}$ is free as an R-module. **Then it is principal.**

Proof: Let $e_1, ..., e_n, ...$ be the generators of $\mathfrak{p}$ freely generating $\mathfrak{p}$. Then $R = \bigoplus_i R \cdot e_i$. However, $e_1 e_2 \in R \cdot e_1 \cap R \cdot e_2$, which is impossible because $R \cdot e_1 \cap R \cdot e_2 = 0$. ■

Dedekind rings and discrete valuation rings

Proposition 4: Let $\mathfrak{m}$ be a maximal ideal of a Dedekind ring R, and $R_{\mathfrak{m}}$ its localization. **Then $R_{\mathfrak{m}}$ is a discrete valuation ring**.

Proof: Every ideal of $R_{\mathfrak{m}}$ is a localization of an ideal from R; localization of a projective module is projective, hence every ideal of $R_{\mathfrak{m}}$ is projective. A projective module over a local ring is free. Therefore, $R_{\mathfrak{m}}$ is a principal ideal ring. ■

COROLLARY: A Dedekind ring R has Krull dimension 1.

Proof: Indeed, any localization of R in a maximal ideal has only two prime ideals. ■

Dedekind rings are Noetherian and integrally closed

LEMMA: Let R be a ring without zero divisors, $\mathfrak{S}$ the set of maximal ideals $\mathfrak{p} \subset R$, and $R_\mathfrak{p}$ the localization of R in $\mathfrak{p}$. **Then $\bigcap_{\mathfrak{p}\in\mathfrak{S}} R_\mathfrak{p} = R$.**

Proof. Step 1: Let $x \in \bigcap_{\mathfrak{p}\in\mathfrak{S}} R_\mathfrak{p}$ and let $I \subset R$ be the set of all $y \in R$ such that $yx \in R$. We are going to prove that $I = R$, which implies that $x \in R$.

Step 2: Arguing ad absurdum, suppose that $I \subsetneq R$. Then there exists a maximal ideal $\mathfrak{m} \supset I$, such that for all $a \notin \mathfrak{m}$, we have $ax \notin R$. Since $x \in R_\mathfrak{m}$, we have $ax \in R$ for some $a \notin \mathfrak{m}$, a contradiction. ■

THEOREM: Any Dedekind ring R is Noetherian and integrally closed.

Proof. Step 1: Any invertible ideal is fractional, hence finitely generated, hence R is Noetherian.

Step 2: Let $\mathfrak{S}$ be the set of prime ideals $\mathfrak{p} \subset R$, and $R_\mathfrak{p}$ is a localization of R in $\mathfrak{p}$. Then $R_\mathfrak{p}$ is DVR (Proposition 4). **For any $x \in k(R)$ finite over R, we have $x \in R_\mathfrak{p}$ for all $\mathfrak{p} \in \mathfrak{S}$, because $R_\mathfrak{p}$ is integrally closed.** Then $x \in \bigcap_{\mathfrak{p}\in\mathfrak{S}} R_\mathfrak{p} = R$ by the previous lemma. ■

Noetherian integrally closed rings are Dedekind

THEOREM: **Any Noetherian integrally closed ring R of Krull dimension 1 is Dedekind.**

Proof. Step 1: Let $I \subset R$ be an ideal. If $II^{-1} \subsetneq R$, **there exists an a prime ideal $\mathfrak{p} \subset R$ containing II^{-1}.** The localization of I in $\mathfrak{p}$ is a principal ideal, generated by $x \in R$, because the localization of R in $\mathfrak{p}$ is DVR. Let b_i be the generators of I. Then $b_i = x\frac{r_i}{s_i}$, where $s_i \notin \mathfrak{p}$.

Step 2: Let $s := \prod s_i$. Then $sx^{-1} \in I^{-1}$, which gives $s \in II^{-1}$. This is a contradiction, because $s_i \notin \mathfrak{p}$. ■

REMARK: Converse is also true: **a Dedekind ring is Noetherian** (because all ideals are invertible, hence finitely generated) and **has Krull dimension 1,** because its localization in a prime ideal is always a DVR, and the localization does not change the Krull dimension.

DEFINITION: Let $[K : \mathbb{Q}]$ be a finite extensions. Then K is called **an algebraic number field**. The integral closure of $\mathbb{Z}$ in K is called **the ring of integers**, denoted $\mathcal{O}_K$.

COROLLARY: The ring of integers in an algebraic number field **is Dedekind.**

Proof: It has the same algebraic dimension as $\mathbb{Z}$, because finite extensions do not change the Krull dimension. ■

Product of prime ideals in a Dedekind ring

Theorem 1: Let R be a ring without zero divisors which is not a field. **Then the following are equivalent.**
(i) R is Dedekind.
(ii) R is Noetherian, and each localization of R at a prime ideal is DVR.
(iii) Every non-zero ideal of R is a product of maximal ideals.

Proof. Step 1: (i) implies (ii) by Proposition 4, and (ii) implies (i) because $R = \bigcap_{\mathfrak{p}\in\mathfrak{S}} R_\mathfrak{p}$, hence (ii) implies that R is integrally closed.

Step 2: To deduce (iii) from (i), take any ideal $I \subset R$. If I is maximal, we are done. If I is not maximal, let $\mathfrak{m}$ be a maximal ideal strictly containing I. Since $\mathfrak{m}^{-1}\mathfrak{m} = R$ and $I \supset \mathfrak{m}$, the product $J := \mathfrak{m}^{-1}I \subset R$ is an ideal. Then $I = \mathfrak{m}J$, and $J \supsetneq I$. Repeating this procedure, we obtain an increasing chain of ideals

$$I \subsetneq \mathfrak{m}_1^{-1} I \subsetneq \mathfrak{m}_1^{-1}\mathfrak{m}_2^{-1} I \subsetneq ...$$

which terminates because R is Noetherian. Then $\mathfrak{m}_1^{-1}\mathfrak{m}_2^{-1}...\mathfrak{m}_n^{-1} I = \mathfrak{m}$ is maximal, giving $I = \mathfrak{m}_1\mathfrak{m}_2...\mathfrak{m}_n\mathfrak{m}$.

Step 3: Let $R_\mathfrak{p}$ be a localization of R in $\mathfrak{p}$. Then every non-zero ideal of $R_\mathfrak{p}$ is a product of non-zero maximal ideals, hence the maximal ideal of $R_\mathfrak{p}$ is principal, and $R_\mathfrak{p}$ is DVR. We proved that (iii) implies (ii). ∎

Factorial Dedekind rings

COROLLARY: A Dedekind ring R is factorial if and only if all ideals in R are principal.

Proof. Step 1: Principal ideal ring is factorial if it is also Noetherian, because **in a Noetherian ring every element can be represented as a product of irreducible ones.**

Step 2: Conversely, let R be a factorial Dedekind ring, $\mathfrak{p} \subset R$ a prime ideal, and $a \in \mathfrak{p}$ a non-zero element. Then $\mathfrak{p}$ contains an irreducible factor t of a. Since R is factorial, the ideal (t) is prime. Since R has Krull dimension 1, the chain $0 \subset \mathfrak{p} \subset (t)$ has length 2, unless $\mathfrak{p} = (t)$. Since every ideal is a product of maximal ideals, every ideal in R is principal. ■

Chinese remainder theorem

DEFINITION: Ideals $P_1, ..., P_n \subset R$ are called **pairwise coprime**, or **comaximal**, if $P_i + P_j = R$ whenever $i \neq j$.

CLAIM: If $P, Q \subset R$ are coprime ideals, **$PQ = P \cap Q$.**

Proof: If $x \in P \cap Q$, and $P + Q = 1$, we have $x \in x(P + Q) \in PQ + QP \subset PQ$, which implies $P \cap Q \subset PQ$.■

PROPOSITION: Let P_1, ..., P_n be comaximal ideals in a ring R. Then $\bigcap_i P_i = P_1 P_2 ... P_n$. Moreover, **Then the natural map $\chi : \frac{R}{\bigcap_i P_i} \longrightarrow \bigoplus_i R/P_i$ is an isomorphsim.**

Proof. Step 1: Let us prove CRT for $n = 2$. **Since P_1, P_2 are coprime, we have $P_1 \cap P_2 = P_1 P_2$.** The map $\chi : \frac{R}{P_1 \cap P_2} \longrightarrow R/P_1 \oplus R/P_2$ is clearly injective. It is surjective, because $1 = p_1 + p_2$, where $p_1 \in P_1, p_2 \in P_2$. This implies $1 - p_1 \in P_2$, which gives $\chi(1 - p_1) = (1, 0)$ and, similarly, $\chi(1 - p_2) = (0, 1)$. Then $\chi(a(1 - p_1) + b(1 - p_2)) = (\underline{a}, \underline{b})$, where $(\underline{a}, \underline{b})$ are classes of $(a, b) \in R^2$ modulo (P_1, P_2).

Chinese remainder theorem (2)

Step 2: We prove CRT using induction in n. By induction assumption, $\prod_{i=1}^{n-1} P_i = \bigcap_{i=1}^{n-1} P_i$ and $\frac{R}{\bigcap_{i=1}^{n-1} P_i} = \bigoplus_{i=1}^{n-1} R/P_i$.

We start by showing that the ideals $\prod_{i=1}^{n-1} P_i$ and P_n are comaximal. Indeed, one has

$$1 = \prod_{i=1}^{n-1} (P_i + P_n) \in \prod_{i=1}^{n-1} P_i + P_n = \bigcap_{i=1}^{n-1} P_i + P_n.$$

Therefore, $\prod_{i=1}^{n-1} P_i = \bigcap_{i=1}^{n-1} P_i$ and P_n are comaximal, and $P_n \prod_{i=1}^{n-1} P_i = P_n \cap \prod_{i=1}^{n-1} P_i = P_n \cap \bigcap_{i=1}^{n-1} P_i = \bigcap_{i=1}^{n} P_i$.

Step 3: $\frac{R}{P_n \cap \bigcap_{i=1}^{n-1} P_i} = \frac{R}{P_n} \bigoplus_{i=1}^{n-1} R/P_i$ because P_n and $\bigcap_{i=1}^{n-1} P_i$ are comaximal. ■

Maximal ideals in Krull dimension 0

LEMMA: Let $\mathfrak{m}$ be a maximal ideal in a ring R without zero divisors. **Then $R/\mathfrak{m}^n$ is local.**

Proof: Let $x \in R$ be an element which does not belong to the maximal ideal, and $y \in R$ an element such that $xy = 1 \mod \mathfrak{m}$. Let w be the image of $xy - 1$ in $R/\mathfrak{m}^n$. Then $w^{n+1} = 0$, which implies $(1-w)(1+w+w^2+...+w^n) = 1$. Therefore, $(1-w)$ is invertible in $R/\mathfrak{m}^n$ Then $x \cdot y(1-w) = 1$, hence x is invertible. ■

Corollary 1: Let $\mathfrak{m} \subset R$ be a maximal ideal, and $R_\mathfrak{m}$ its localization. **Then $R_\mathfrak{m}/\mathfrak{m}^n = R/\mathfrak{m}^n$.** ■

Ideals in a Dedekind ring are generated by two elements

THEOREM: Let R be a Dedekind ring, and $I \subset R$ an ideal. Then **for any ideal $J \supset I$, there exists $b \in R$ such that $J = I + (b)$.** Moreover, **I can be generated by two elements of R.**

Proof. Step 1: Let $\mathfrak{p}_i$ non-zero prime ideals, and $I = \prod_i \mathfrak{p}_i^{\alpha_i}$. Since R has Krull dimension 1, the ideals $\mathfrak{p}_1$, $\mathfrak{p}_2$,..., are pairwise comaximal. Since $\sqrt{\mathfrak{p}_i^n} = \mathfrak{p}_i$, one can never have $\mathfrak{p}_i^n \subset \mathfrak{p}_j$. This implies that **the ideals $\mathfrak{p}_i^{\alpha_n}$ are also pairwise comaximal.**

Step 2: Chinese Remainder Theorem implies that $R/I = \bigoplus_i \frac{R}{\mathfrak{p}_i^{\alpha_i}}$. By Corollary 1, each of the factors $\frac{R}{\mathfrak{p}_i^{\alpha_i}}$ is a quotient of DVR, hence **any ideal in $\frac{R}{\mathfrak{p}_i^{\alpha_i}}$ and in R/I is principal.** This implies that $J = I + (b)$.

Step 3: If $a \in I$ is non-zero, we apply Step 2 to the ideals $(a) \subset I$, and find that $I = (a) + (b)$. ■

Uniqueness of the prime ideal decomposition in a Dedekind ring

THEOREM: Let R be a Dedekind ring, $I \subset R$ an ideal, $\mathfrak{p}_i$ non-zero prime ideals, and $I = \prod_i \mathfrak{p}_i^{\alpha_i}$ its prime decomposition. Then **this decomposition is unique, up to renumbering the prime factors.**

Proof: Let $I = \prod_j \mathfrak{q}_j^{\beta_j}$ be another such decomposition. If there is a common factor, say, $\mathfrak{p}$, between $\mathfrak{p}_i$ and $\mathfrak{q}_i$, we can replace I by $I\mathfrak{p}^{-1}$, and have two non-equivalent decompositions with less factors. Repeating this procedure, we obtain that $I = \prod_i \mathfrak{p}_i^{\alpha_i} = \prod_j \mathfrak{q}_j^{\beta_j}$. However, the ideals $\mathfrak{q}_i^{\alpha_i}$ and $\mathfrak{q}_j^{\beta_j}$ are pairwise comaximal, which implies (by Step 2 of the proof of CRT given above) that $I = \prod_i \mathfrak{p}_i^{\alpha_i} \cap \prod_j \mathfrak{q}_j^{\beta_j} = \prod_i \mathfrak{p}_i^{\alpha_i} \prod_j \mathfrak{q}_j^{\beta_j} = I^2$, which is impossible by Krull theorem. ■

Dedekind rings: equivalent conditions

THEOREM: Let R be a ring without zero divisors. **Then the following are equivalent:**

(i) R is Dedekind (that is, all ideals are invertible as fractional ideals).

(ii) R is Noetherian and any localization of R in a prime ideal is DVR

(iii) Every non-zero ideal in R is equal to a product of maximal ideals

(iv) R is integrally closed, Noetherian ring of Krull dimension 1

(v) Every ideal in R is projective as an R-module.

Proof. Step 1: The equivalence (v) $\Rightarrow$ (i) is clear because finitely generated projective fractional ideals are the same as invertible (Lecture 14). The implication (i) $\Rightarrow$ (ii) is also clear because invertibility of the ideal implies that it is finitely generated projective, after passing to the localization, all finitely generated projective modules are free, hence the localization is a principal ideal ring, therefore DVR. The implicationsequivalences (ii) $\Leftrightarrow$ (i) $\Leftrightarrow$ (iii) and the equivalence (iv) $\Leftrightarrow$ (ii) were proven earlier today. ∎