

# Commutative Algebra

## lecture 17: Smooth points in algebraic varieties

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## Topological manifolds

**REMARK:** Manifolds can be smooth (of a given “differentiability class”), real analytic, or topological (continuous).

**DEFINITION:** **Topological manifold** is a topological space which is locally homeomorphic to an open ball in  $\mathbb{R}^n$ .

**EXERCISE:** Show that a group of homeomorphisms acts on a connected manifold transitively.

**DEFINITION:** Such a topological space is called **homogeneous**.

**Open problem:** (Busemann)

**Characterize manifolds among other homogeneous topological spaces.**

## Banach fixed point theorem

### LEMMA: (Banach fixed point theorem/ “contraction principle”)

Let  $U \subset \mathbb{R}^n$  be a closed subset, and  $f : U \rightarrow U$  a map which satisfies  $|f(x) - f(y)| < k|x - y|$ , where  $k < 1$  is a real number (such a map is called “contraction”). **Then  $f$  has a fixed point, which is unique.**

**Proof. Step 1:** Uniqueness is clear because for two fixed points  $x_1$  and  $x_2$   $|f(x_1) - f(x_2)| = |x_1 - x_2| < k|x_1 - x_2|$ .

**Step 2:** Existence follows because the sequence  $x_0 = x, x_1 = f(x), x_2 = f(f(x)), \dots$  satisfies  $|x_i - x_{i+1}| \leq k|x_{i-1} - x_i|$  which gives  $|x_n - x_{n+1}| < k^n a$ , where  $a = |x - f(x)|$ . Then  $|x_n - x_{n+m}| < \sum_{i=0}^m k^{n+i} a \leq k^n \frac{1}{1-k} a$ , hence  $\{x_i\}$  is a Cauchy sequence, and converges to a limit  $y$ , which is unique.

**Step 3:**  $f(y)$  is a limit of a sequence  $f(x_0), f(x_1), \dots, f(x_i), \dots$  which gives  $y = f(y)$ . ■

**EXERCISE:** Find a counterexample to this statement when  $U$  is open and not closed.

## Differentiable maps

**DEFINITION:** Let  $U, V \subset \mathbb{R}^n$  be open subsets. **An affine map** is a sum of linear map  $\alpha$  and a constant map. Its **linear part** is  $\alpha$ .

**DEFINITION:** Let  $U \subset \mathbb{R}^m, V \subset \mathbb{R}^n$  be open subsets. A map  $f : U \rightarrow V$  is called **differentiable** if it can be approximated by an affine one at any point: that is, for any  $x \in U$ , there exists an affine map  $\varphi_x : \mathbb{R}^m \rightarrow \mathbb{R}^n$  such that

$$\lim_{x_1 \rightarrow x} \frac{|f(x_1) - \varphi(x_1)|}{|x - x_1|} = 0$$

**DEFINITION: Differential**, or **derivative** of a differentiable map  $f : U \rightarrow V$  is the linear part of  $\varphi$ .

**DEFINITION: Diffeomorphism** is a differentiable map  $f$  which is invertible, and such that  $f^{-1}$  is also differentiable. A map  $f : U \rightarrow V$  is a **local diffeomorphism** if each point  $x \in U$  has an open neighbourhood  $U_1 \ni x$  such that  $f : U_1 \rightarrow f(U_1)$  is a diffeomorphism.

**REMARK: Chain rule** says that a composition of two differentiable functions is differentiable, and its differential is composition of their differentials.

**REMARK:** Chain rule implies that **differential of a diffeomorphism is invertible**. Converse is also true (in a sense):

## Inverse function theorem

**THEOREM:** Let  $U, V \subset \mathbb{R}^n$  be open subsets, and  $f : U \rightarrow V$  a differentiable map. Suppose that the differential of  $f$  is everywhere invertible. **Then  $f$  is locally a diffeomorphism.**

**Proof. Step 1:** Let  $x \in U$ . Without restricting generality, we may assume that  $x = 0$ ,  $U = B_r(0)$  is an open ball of radius  $r$ , and **in  $U$  one has  $\frac{|f(x_1) - \varphi(x_1)|}{|x - x_1|} < 1/2$ .** Replacing  $f$  with  $-f \circ (D_0 f)^{-1}$ , where  $D_0 f$  is differential of  $f$  in 0, **we may assume also that  $D_0 f = -\text{Id}$ .**

**Step 2:** In these assumptions,  $|f(x) + x| < 1/2|x|$ , hence  $\psi_s(x) := f(x) + x - s$  is a contraction. This map maps  $\overline{B}_{r/2}(0)$  to itself when  $s < r/4$ . By Banach fixed point theorem,  **$\psi_s(x) = x$  has a unique fixed point  $x_s$ , which is obtained as a solution of the equation  $f(x) + x - s = x$ , or, equivalently,  $f(x) = s$ .** Denote the map  $s \rightarrow x_s$  by  $g$ .

**Step 3:** By construction,  $fg = \text{Id}$ . Applying the chain rule again, we find that  $g$  is also differentiable. ■

**REMARK:** Usually, diffeomorphisms are assumed **smooth** (infinitely differentiable). **A smooth version of this result is left as an exercise.**

## Critical points and critical values

**DEFINITION:** Let  $U \subset \mathbb{R}^m, V \subset \mathbb{R}^n$  be open subsets, and  $f : U \rightarrow V$  a smooth function. A point  $x \in U$  is a **critical point** of  $f$  if the differential  $D_x f : \mathbb{R}^m \rightarrow \mathbb{R}^n$  is not surjective. **Critical value** is an image of a critical point. **Regular value** is a point of  $V$  which is not a critical value.

**THEOREM: (Sard's lemma)**

**The set of critical values of  $f$  is of measure 0 in  $V$ .**

**REMARK:** We leave this theorem without a proof. We won't use it.

**DEFINITION:** A subset  $M \subset \mathbb{R}^n$  is an  **$m$ -dimensional smooth submanifold** if for each  $x \in M$  there exists a neighbourhood  $U \subset \mathbb{R}^n$  containing  $x$  and a diffeomorphism from  $U$  to an open ball  $B \subset \mathbb{R}^n$  which maps  $U \cap M$  to an intersection  $B \cap \mathbb{R}^m$  of  $B$  and an  $m$ -dimensional linear subspace.

**REMARK:** Clearly, **a smooth submanifold is a (topological) manifold.**

**THEOREM:** Let  $U \subset \mathbb{R}^m, V \subset \mathbb{R}^n$  be open subsets,  $f : U \rightarrow V$  a smooth function, and  $y \in V$  a regular value of  $f$ . **Then  $f^{-1}(y)$  is a smooth submanifold of  $U$ .**

## Preimage of a regular value

**THEOREM:** Let  $U \subset \mathbb{R}^m, V \subset \mathbb{R}^n$  be open subsets,  $f : U \rightarrow V$  a smooth function, and  $y \in V$  a regular value of  $f$ . **Then  $f^{-1}(y)$  is a smooth submanifold of  $U$ .**

**Proof:** Let  $x \in U$  be a point in  $f^{-1}(y)$ . It suffices to prove that  $x$  has a neighbourhood diffeomorphic to an open ball  $B$ , such that  $f^{-1}(y)$  corresponds to a linear subspace in  $B$ . Without restricting generality, we may assume that  $y = 0$  and  $x = 0$ .

The differential  $D_0f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is surjective. Let  $L := \ker D_0f$ , and let  $A : \mathbb{R}^n \rightarrow L$  be any map which acts on  $L$  as identity. Then  $D_0f \oplus A : \mathbb{R}^n \rightarrow \mathbb{R}^m \oplus L$  is an isomorphism of vector spaces. **Therefore,  $\psi : f \oplus A$  mapping  $x_1$  to  $f(x_1) \oplus A(x_1)$  is a diffeomorphism in a neighbourhood of  $x$ .** However,  $f^{-1}(0) = \psi^{-1}(0 \oplus L)$ . We have constructed a diffeomorphism of a neighbourhood of  $x$  with an open ball mapping  $f^{-1}(0)$  to  $0 \oplus L$ . ■

## Regular maps and solutions of a set of equations

**COROLLARY:** Let  $f_1, \dots, f_m$  be smooth functions on  $U \subset \mathbb{R}^n$  such that the differentials  $df_i$  are linearly independent everywhere. **Then the set of solutions of equations  $f_1(z) = f_2(z) = \dots = f_m(z) = 0$  is a smooth  $(n-m)$ -dimensional submanifold in  $U$ .**

**DEFINITION: Smooth hypersurface** is a closed codimension 1 submanifold.

**EXERCISE:** Prove that **a smooth hypersurface in  $U$  is always obtained as a solution of an equation  $f(z) = 0$** , where 0 is a regular value of a function  $f : U \rightarrow \mathbb{R}$ .

## Abstract manifolds: charts and atlases

**DEFINITION:** An **open cover** of a topological space  $X$  is a family of open sets  $\{U_i\}$  such that  $\bigcup_i U_i = X$ . A cover  $\{V_i\}$  is a **refinement** of a cover  $\{U_i\}$  if every  $V_i$  is contained in some  $U_i$ .

**REMARK:** Any two covers  $\{U_i\}, \{V_i\}$  of a topological space admit a **common refinement**  $\{U_i \cap V_j\}$ .

**DEFINITION:** Let  $M$  be a topological manifold. A cover  $\{U_i\}$  of  $M$  is an **atlas** if for every  $U_i$ , we have a map  $\varphi_i : U_i \rightarrow \mathbb{R}^n$  giving a homeomorphism of  $U_i$  with an open subset in  $\mathbb{R}^n$ . In this case, one defines the **transition maps**

$$\Phi_{ij} : \varphi_i(U_i \cap U_j) \rightarrow \varphi_j(U_i \cap U_j)$$

**DEFINITION:** A function  $\mathbb{R} \rightarrow \mathbb{R}$  is of **differentiability class  $C^i$**  if it is  $i$  times differentiable, and its  $i$ -th derivative is continuous. A map  $\mathbb{R}^n \rightarrow \mathbb{R}^m$  is of **differentiability class  $C^i$**  if all its coordinate components are. A **smooth function/map** is a function/map of class  $C^\infty = \bigcap C^i$ .

**DEFINITION:** An atlas is **smooth** if all transition maps are smooth (of class  $C^\infty$ , i.e., infinitely differentiable), **smooth of class  $C^i$**  if all transition functions are of differentiability class  $C^i$ , and **real analytic** if all transition maps admit a Taylor expansion at each point.

## Covering maps

**DEFINITION:** Let  $\varphi : \tilde{M} \rightarrow M$  be a continuous map of manifolds (or CW complexes). We say that  $\varphi$  is **a covering** if  $\varphi$  is locally a homeomorphism, and for any  $x \in M$  there exists a neighbourhood  $U \ni x$  such that  $\varphi^{-1}(U)$  is a disconnected union of several manifolds  $U_i$  such that the restriction  $\varphi|_{U_i}$  is a homeomorphism from  $U_i$  to  $U$ .

**DEFINITION:** Let  $\Gamma$  be a discrete group continuously acting on a locally compact topological space  $M$ . This action is called **properly discontinuous** if the space of orbits of  $\Gamma$  is Hausdorff.

**THEOREM:** Let  $\Gamma$  be a discrete group acting on  $M$  freely and properly discontinuously. **Then  $M \rightarrow M/\Gamma$  is a covering.**

This result is left as an exercise.

## Finite coverings

**EXAMPLE:** A map  $x \rightarrow nx$  in a circle  $S^1$  is a covering.

**EXAMPLE:** For any non-degenerate integer matrix  $A \in \text{End}(\mathbb{Z}^n)$ , the corresponding map of a torus  $T^n$  is a covering.

**CLAIM:** Let  $\varphi : \tilde{M} \rightarrow M$  be a covering, with  $M$  connected. **Then the number of preimages  $|\varphi^{-1}(m)|$  is constant in  $M$ .**

**Proof:** Since  $\varphi^{-1}(U)$  is a disconnected union of several copies of  $U$ , this number is a locally constant function of  $m$ . ■

**DEFINITION:** Let  $\varphi : \tilde{M} \rightarrow M$  be a covering, with  $M$  connected. Assume that  $\tilde{M}, M$  are oriented and  $\varphi$  preserves the orientation. The number  $|\varphi^{-1}(m)|$  is called **degree** of a map  $\varphi$ .

**CLAIM:** Any covering  $\varphi : \tilde{M} \rightarrow M$  with  $\tilde{M}$  compact **has finite degree.**

**Proof:** Take  $U$  in such a way that  $\varphi^{-1}(U)$  is a disconnected union of several copies of  $U$ , and let  $x \in U$ . Then  $\varphi^{-1}(x)$  is discrete, and since  $\tilde{M}$  is compact, any discrete subset of  $\tilde{M}$  is finite. ■

## Finite projection maps (reminder)

**PROPOSITION:** Let  $X \subset \mathbb{C}^n$  be an irreducible affine subvariety,  $z_i$  coordinates on  $\mathbb{C}^n$ , and  $z_1, \dots, z_k$  transcendence basis on  $k(X)$ . Then, for all  $\lambda_1, \dots, \lambda_k$  outside of the zero-set of a certain non-zero homogeneous polynomial, **the function  $z_n \in \mathcal{O}_X$  is a root of a monic polynomial in the variables  $z'_1, \dots, z'_k$ , where  $z'_i := z_i + \lambda_i z_n$ .**

**Proof:** Lecture 15. ■

### Corollary 1: (Noether's normalization lemma, first version)

Let  $X \subset \mathbb{C}^n$  be an irreducible affine subvariety,  $z_i$  coordinates on  $\mathbb{C}^n$ , and  $z_1, \dots, z_k$  transcendence basis on  $k(X)$ . Then **there exists a linear coordinate change  $z'_i := z_i + \sum_{j=1}^{n-k} \lambda_{j+k} z_{j+k}$ , such that the projection  $\Pi_k : X \rightarrow \mathbb{C}^k$  to the first  $k$  arguments is a finite, dominant morphism.**

**Proof:** Previous proposition shows that the projection  $P_n : X \rightarrow \mathbb{C}^{n-1}$  is finite onto its image  $X_1$  (after some linear adjustment). Using induction by  $n$ , we can assume that  $P_k : X_1 \rightarrow \mathbb{C}^k$  is also finite, hence the composition map is finite **(composition of finite morphisms is always finite, as we have seen).** ■

## Multi-valued functions

**DEFINITION:** Define **a complex manifold** as a manifold equipped with an atlas  $\{U_i\}$ , with each open subset  $U_i \subset M$  identified with an open ball in  $\mathbb{C}^n$ , and complex analytic transition functions.

**DEFINITION:** Define **a complex variety** as a subvariety  $Z \subset M$  in a complex manifold given by a collection of complex analytic equations.

**DEFINITION: Multi-valued function** on  $M$  is a closed, irreducible complex subvariety  $Z \subset M \times \mathbb{C}$  such that the projection  $Z \rightarrow M$  is locally a diffeomorphism outside of a closed, nowhere dense subset in  $Z$ . The set  $Z$  is called **the graph of the multi-valued function**.

**EXAMPLE: Logarithm is a multi-valued function** on  $\mathbb{C}$ . Indeed, let  $Z$  be the graph of exponent  $y = e^x$  in  $\mathbb{C}^2$ . The projection to  $x$  expresses all branches of logarithm  $x = \log y$  as functions of  $y$ .

**EXAMPLE:  $y \rightarrow \sqrt{y}$  is a multi-valued function.** Indeed, the graph of  $y = x^2$  projected to  $x$  gives both branches of  $\sqrt{y}$ .

## Multi-valued functions and branched covers

**THEOREM:** Consider a subvariety  $Z \subset \mathbb{C}^{n+1}$  given by a monic polynomial equation  $P(t) = 0$ , with  $P(t) \in \mathcal{O}_{\mathbb{C}^n}[t]$ . Assume that  $P(t)$  is irreducible. **Then  $Z$  is a graph of a multi-valued function.** Moreover, the projection of  $Z$  to  $\mathbb{C}^n$  (to the first  $n$  coordinates) is a diffeomorphism at  $(z, t)$  if and only if  $P'(z) \neq 0$ .

**Proof. Step 1:** We shall represent points of  $\mathbb{C}^{n+1}$  by pairs  $(z, t)$ , with  $z = (z_1, \dots, z_n)$ . Let  $\pi : Z \rightarrow \mathbb{C}^n$  be the standard projection along  $t$ . By the inverse function theorem,  **$Z \subset \mathbb{C}^{n+1}$  is a smooth submanifold in a neighbourhood of any point  $(z, t) \in Z$ , with  $z \in \mathbb{C}^n$  whenever the differential  $dP : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$  is surjective (non-zero) at  $(z, t)$ .**

**Step 2:** This implies that  $Z$  is smooth outside of the set of all  $(z, t) \in \mathbb{C}^{n+1}$  such that  $dP(z, t)(\xi, \tau) = 0$ , for all  $\xi \in \langle d/dz_1, \dots, d/dz_n \rangle$ , and  $\tau \in \langle d/dt \rangle$ . Let  $P(z, t) = t^k + \sum_{i=0}^{k-1} t^i a_i(z)$ . Then

$$dP(z, t) = kt^{k-1}dt + \sum_{i=0}^{k-1} t^i da_i(z) + \sum_{i=0}^{k-1} it^{i-1} a_i(z) dt.$$

**For  $|t| \gg 0$ , the leading term  $kt^{k-1}dt + t^{k-1}da_{k-1}(z)$  dominates the rest, and it is non-zero, because its  $dt$  component is non-zero.**

## Multi-valued functions and branched covers (2)

**THEOREM:** Consider a subvariety  $Z \subset \mathbb{C}^{n+1}$  given by a monic polynomial equation  $P(t) = 0$ , with  $P(t) \in \mathcal{O}_{\mathbb{C}^n}[t]$ . Assume that  $P(t)$  is irreducible. **Then  $Z$  is a graph of a multi-valued function.** Moreover,  $Z$  is smooth, and the projection of  $Z$  to  $\mathbb{C}^n$  (to the first  $n$  coordinates) is a local diffeomorphism at  $(z, t)$  if and only if  $P'(z) \neq 0$ .

**Step 3:** Let  $z \in Z$  be a smooth point, and  $(\xi, \tau) \in T_z Z$ . Then  $\pi : W \rightarrow \mathbb{C}^n$  is invertible whenever  $W$  does not contain a vector  $(0, \tau)$ , equivalently, when  $dP(z, t)(0, \tau) \neq 0$ . **This is equivalent to  $\frac{dP(z, t)}{dt} \neq 0$ .**

**Step 4:** Let  $P'(z, t) := \frac{dP(z, t)}{dt}$ . To prove that  $Z$  defines a multi-valued function, it remains to show that  $P'(z, t)$  is not identically zero on  $Z$ . Since  $P(z, t)$  is irreducible, and  $\mathcal{O}_{\mathbb{C}^{n+1}}$  is factorial, the ring  $\frac{\mathcal{O}_{\mathbb{C}^{n+1}}}{(P)}$  has no zero divisors. Then Hilbert Nullstellensatz would imply that any function  $f \in \mathcal{O}_{\mathbb{C}^{n+1}}$  which vanishes on  $Z$  is divisible by  $P(z, t)$ . Then  $P'(z, t)$  does not vanish on  $Z$ , because it is a polynomial of smaller degree. ■

## Symmetric polynomials

**DEFINITION: Symmetric polynomial**  $P(z_1, \dots, z_n) \in \mathbb{C}[z_1, \dots, z_n]$  is a polynomial which is invariant with respect to the symmetric group  $\Sigma_n$  acting on  $\mathbb{C}[z_1, \dots, z_n]$  in a usual way.

**DEFINITION:** Consider the polynomial  $P(z_1, \dots, z_n, t) := \prod_{i=1}^n (t + z_i) = \sum e_i t^i$ , with  $e_i \in \mathbb{C}[z_1, \dots, z_n]$ . Then  $e_i$  are called **elementary symmetric polynomials** on  $z_1, \dots, z_n$ .

**THEOREM:** Every symmetric polynomial on  $z_1, \dots, z_n$  can be polynomially expressed through the elementary symmetric polynomials.

**Proof:** Left as an exercise. ■

## Discriminant of a polynomial

**DEFINITION:** Consider the symmetric polynomial  $\prod_{i \neq j} (z_i - z_j)$ . **Discriminant** of the polynomial  $P(z_1, \dots, z_n, t) := \prod_{i=1}^n (t - z_i)$  is  $\prod_{i \neq j} (z_i - z_j)$  considered as a polynomial of its coefficients.

**EXAMPLE: Discriminant of the quadratic polynomial  $t^2 + bt + c$  is  $b^2 - 4c$ .**

**EXAMPLE: Discriminant of the cubic polynomial  $t^3 + bt^2 + ct + d$  is  $b^2c^2 - 4c^3 - 4b^3d - 27d^2 + 18bcd$ .**

**CLAIM: A polynomial has no multiple roots if and only if its discriminant is non-zero. ■**

**Corollary 1:** Let  $P(t) \in k[t]$  be a polynomial over an algebraically closed field, and  $D$  its discriminant. **Then the derivative  $P'(t)$  does not vanish on all roots of  $t$  if and only if  $D \neq 0$ .**

**Proof:** Let  $\alpha$  be a root of  $P$ . Then  $P(t) = P_\alpha(t)(t - \alpha)$ , and  $P'(t) = P'_\alpha(t)(t - \alpha) + P_\alpha(t)$ . Therefore,  $P'(\alpha) = 0$  if and only if  $P_\alpha(\alpha) = 0$ . This is equivalent to  $P(t)$  being divisible by  $(t - \alpha)^2$ . ■

## Discriminant and ramified coverings

**THEOREM:** Consider a subvariety  $Z \subset \mathbb{C}^{n+1}$  given by an irreducible monic polynomial equation  $P(t) = 0$  of degree  $k$ , with  $P(t) \in \mathbb{C}[z_1, \dots, z_n][t]$ , and let  $\pi : Z \rightarrow \mathbb{C}^n$  be the projection to the coordinates  $z_1, \dots, z_n$ . Assume that  $P(t)$  is irreducible. Denote by  $D(z)$  the discriminant of  $P(t)$ , considered as a polynomial function on  $(z_1, \dots, z_n)$ , and let  $U \subset \mathbb{C}^n$  be the set of all  $z \in \mathbb{C}^n$  such that  $D(z) \neq 0$ . **Then the intersection  $Z \cap \pi^{-1}(U)$  is smooth and the projection  $\pi : Z \cap \pi^{-1}(U) \rightarrow U$  is a  $d$ -sheeted covering.**

**Proof. Step 1:** Since  $P(t)$  is irreducible, it has no common divisors with  $P'(t)$ . Since the polynomial ring is factorial, irreducibility implies that the ideal  $(P(t))$  is prime, hence  $P'(t)$  does not identically vanish on  $Z$ . Therefore  $U$  is open and nowhere dense. By Corollary 1, for any  $z \in U$ , the polynomials  $P(z, t)$  and  $P'(z, t)$  have no common roots. Therefore,  $\frac{dP(z, t)}{dt} \neq 0$  on  $Z \cap \pi^{-1}(U)$ , and **the set  $Z = \{(z, t) \mid P(z, t) = 0\}$  is smooth outside of zeros of  $D(z)$ .**

**Step 2:** Let  $z \in Z \cap \pi^{-1}(U)$  and  $(\xi, \tau) \in T_{(z, t)}Z$ . Then  $\pi : T_{(z, t)}Z \rightarrow \mathbb{C}^n$  is invertible whenever  $T_{(z, t)}$  does not contain a vector  $(0, \tau)$ , equivalently, when  $dP(z, t)(0, \tau) \neq 0$ . **This is equivalent to  $\frac{dP(z, t)}{dt} \neq 0$ .**

**Step 3:** The map  $\pi : Z \cap \pi^{-1}(U) \rightarrow U$  is locally a diffeomorphism, and each point has precisely  $d$  preimages. **Therefore, it is a covering. ■**

## Every algebraic variety is a ramified cover

Comparing this with the Noether normalization lemma, we obtain the following theorem.

**COROLLARY:** Let  $X$  be an algebraic variety. Then there exists a birational, finite map  $X \rightarrow Z$ , a divisor  $D \subset Z$ , and a divisor  $D_1 \subset \mathbb{C}^n$ , **such that  $Z \setminus D$  is a  $d$ -sheeted covering of  $\mathbb{C}^n \setminus D_1$ .** ■

**COROLLARY:** **Every algebraic variety  $X$  over  $\mathbb{C}$  has a smooth point.** Moreover, **non-smooth points of  $X$  are contained in a proper algebraic subvariety of  $X$ .**

**Proof:** Indeed, **every birational map is an isomorphism outside of a divisor.** ■