Commutative Algebra

lecture 18: Hilbert polynomial

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Associated graded ring

DEFINITION: A **multiplicative filtration** on a ring A is a sequence $A = F_0 \supset F_1 \supset ...$ such that all F_i are closed under multiplication and satisfy $F_i F_j \subset F_{i+j}$. A ring equipped with a multiplicative filtration is called **filtered**. An **associated graded quotient** of a filtered ring is $\oplus_{i=0}^\infty A^i$, where $A^i = F_i/F_{i+1}$.

CLAIM: $A = F_0 \supset F_1 \supset F_2 \supset ...$ be a filtered ring, $a_1, a_2 \in F_i$ and $b_1, b_2 \in F_j$. Assume that $a_1 = a_2 \mod F_{i+1}$ and $b_1 = b_2 \mod F_{j+1}$. **Then** $a_1 b_1 = a_2 b_2 \mod F_{i+1}F_j + F_i F_{j+1}$.

Proof: $a_1 b_1 - a_1 b_2 = a_1(b_1 - b_2) = 0 \mod F_i F_{j+1}$ and
$a_1 b_2 - a_2 b_2 = (a_1 - a_2) b_2 = 0 \mod F_{i+1} F_j$. ■

REMARK: Since $F_{i+1}F_j + F_i F_{j+1} \subset F_{i+j+1}$, the product of $a \in F_i/F_{i+1}$ and $b \in F_j/F_{j+1}$ is well defined as an element of F_{i+j}/F_{i+j+1}. Therefore, **the associated graded ring $\oplus_{i=0}^\infty A^i$ is equipped with a natural ring structure.** This ring is called **the associated graded ring**.

EXAMPLE: Let $I \subset A$ be an ideal. Then $A \supset I \supset I^2 \supset ...$ is a multiplicative filtration. The corresponding associated graded ring $A^* := \bigoplus_{i=0}^\infty \frac{I^i}{I^{i+1}}$ is called **the associated graded ring of the ideal I.**

Finitely generated modules over a graded ring

CLAIM: The associated graded ring of a Noetherian local ring **is finitely generated.** ■

DEFINITION: Let $A^* = \bigoplus_{i=0}^\infty A^i$ be a graded ring. **Graded module over a graded ring** is a module $M^* = \bigoplus_{i=0}^\infty M^i$ such that $A^i M^j \subset M^{i+j}$.

CLAIM: Let M^* be a finitely generated graded module over a graded ring A^*, with all A^i finite-dimensional over a field $A^0 = k$. **Then** $\dim_k M^i < \infty$. ■

EXERCISE: Let $f : \mathbb{Z}^{\geqslant 0} \longrightarrow \mathbb{Z}$ be a function. Assume that $g(n) := f(n+1) - f(n)$ is polynomial for $n \gg 0$. **Prove that $f(n)$ is polynomial for $n \gg 0$.**

DEFINITION: Let M^* be a finitely generated graded module over a graded ring A^*, with $A^0 = k$ a field. The **Hilbert function** of M^* is $h_M(n) := \dim_k M^n$.

THEOREM: Let M^* be a finitely generated graded module over a finite-generated graded ring A^*, generated by A^1, with $A^0 = k$ a field. **Then the Hilbert function of $h_M(n)$ is polynomial for n sufficiently big.**

Proof: Next slide

Hilbert polynomial of a graded module

THEOREM 1: Let M^* be a finitely generated graded module over a graded ring A^* generated by a finite-dimensional space A^1, with $A^0 = k$ a field. **Then the Hilbert function of $h_M(n)$ is polynomial for n sufficiently big.**

Proof. Step 1: Let A^* be a finitely generated graded ring, and

$$0 \longrightarrow M_1^* \longrightarrow M_2^* \longrightarrow M_3^* \longrightarrow 0.$$

an exact sequence of finitely generated graded A^*-modules. Then $h_{M_2}(n) = h_{M_1}(n) + h_{M_3}(n)$. Therefore, **the Hilbert function is polynomial for M_i if it is polynomial for M_j, M_k.**

Hilbert polynomial of a graded module (2)

Step 2: Let A^* be a finitely generated graded ring, M^* a torsion-free A^*-module, and $a \in A^1$. Denote the map $m \mapsto ma$ by L_a. Consider an exact sequence $0 \longrightarrow \ker L_a \longrightarrow M^* \xrightarrow{L_a} M^{*+1} \longrightarrow \frac{M^{*+1}}{aM^*} \longrightarrow 0$. Let $U := \frac{M^{*+1}}{aM^*}$. **Suppose that the Hilbert function $h_U(n)$ and $h_{\ker L_a}(n)$ is polynomial for $n \gg 0$. Since $h_M(n+1) - h_M(n) = h_U(n) - h_{\ker L_a}$, this implies that $h_M(n)$ is polynomial for $n \gg 0$** (Exercise 1).

Step 3: For any graded, finitely generated k-module, $h_M(n) = 0$ for $n \gg 0$. **This proves Theorem 1 for $A^* = k$.**

Step 4: Let A^* be a graded ring, generated by d elements in A^1. **We prove that $h_M(n)$ is polynomial using induction in d.** For $d = 1$ it follows from Step 3. Using induction in d, we assume that for any graded ring B^* generated by d elements, and any finitely-generated B^*-module U^*, the Hilbert function $h_U(n)$ is polynomial for $n \gg 0$.

Step 5: Let $a \in A^k$ be a generator of A^*, and
$0 \longrightarrow \ker L_a \longrightarrow M^* \xrightarrow{L_a} M^{*+1} \longrightarrow \frac{A^{*+1}}{aA^*} \longrightarrow 0$ the exact sequence of Step 2. By induction assumption, the Hilbert function of $\frac{A^{*+1}}{aA^*}$-modules is polynomial for $n \gg 0$. Since $\ker L_a$ is a module over $\frac{A^{*+1}}{aA^*}$, the Hilbert function $h_{\ker L_a}(n)$ is polynomial. **By Step 2, the $H_M(n)$ is also polynomial.** ■

Krull dimension and the degree of the Hilbert polynomial

Step 2 of Theorem 1 also brings the following corollary

COROLLARY: Let M be a graded A^*-module and $a \in A^k$ an element such that the multiplication map $L_a: M \longrightarrow M$, $L_a(x) = ax$ is injective. Denote by N the module M/aM. **Then** $\deg h_N(n) = \deg h_M(n) - 1$. ■

THEOREM 2: Let R be a Noetherian local ring without zero divisors, $\mathfrak{m}$ its maximal ideal, $A := \bigoplus_{i=0}^{\infty} \frac{\mathfrak{m}^i}{\mathfrak{m}^{i+1}}$ its associated graded ring, and $h_A(n) := \dim A^n$. **Then the Hilbert function $h_A(n)$ is a polynomial for $n \gg 0$.** Moreover, **its degree d is equal to the Krull dimension of R.**

Proof. Step 1: By Nakayama lemma, A^1 is 1-dimensional and generates A^*. **Then $h_A(n)$ is polynomial by Theorem 1.**

Step 2: Let $\tilde{R}$ be an integral closure of R. By Cohen-Seidenberg, the Krull dimension of $\tilde{R}$ is equal to the Krull dimension of R. Denote by $\tilde{A}$ the associated graded ring of $\tilde{R}$. Then $\deg h_A(n) = \deg h_{\tilde{A}}(n)$, because $\tilde{A}$ is a finite extension of A. **Replacing R by $\tilde{R}$ if necessary, we may assume that R is normal.**

Krull dimension and the degree of the Hilbert polynomial (2)

THEOREM 2: Let R be a Noetherian local ring without zero divisors, $\mathfrak{m}$ its maximal ideal, $A := \bigoplus_{i=0}^{\infty} \frac{\mathfrak{m}^i}{\mathfrak{m}^{i+1}}$ its associated graded ring, and $h_A(n) := \dim A^n$. **Then the Hilbert function $h_A(n)$ is a polynomial for $n \gg 0$. Moreover, its degree d is equal to the Krull dimension of R.**

Step 2: We can assume that R is normal.

Step 3: Let $0 \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \subsetneq ... \subsetneq \mathfrak{p}_n \subsetneq R$ be a chain of prime ideals of maximal possible length. Since $R_{\mathfrak{p}_1}$ is a local integrally closed ring of Krull dimension 1, it is a discrete valuation ring, and its maximal ideal is principal. Therefore, **in the ring $R_{\mathfrak{p}_1}$ we have $\mathfrak{p}_1 = (p_1)$, where $p_1 \in \mathfrak{p}_1$.**

Step 4: Choose $p \in \mathfrak{p}_1$ which is mapped to p_1 after the localization. The localization of $R/(p)$ in $\mathfrak{p}_1$ is a field, hence $\mathfrak{p}_1$ is a minimal prime ideal in $R/(p)$. Therefore the chain of prime ideals $\mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \subsetneq ... \subsetneq \mathfrak{p}_n \subsetneq R/(p)$ is maximal, and $R/(p)$ has Krull dimension $\dim R - 1$.

Step 5: Let $a \in A^1 = \frac{\mathfrak{m}^1}{\mathfrak{m}^2}$ be the class represented by p. Using induction in $\dim R$, we may assume that $\deg h_{A/(a)}(n) = \dim R/(p) = \dim R - 1$. By Theorem 1, Step 2, $\deg h_{A/(a)}(n) = \deg h_A(n) - 1$. This gives $\deg h_A(n) = \deg h_{A/(a)}(n) + 1 = \dim R/(p) + 1 = \dim R$. ■