Commutative Algebra

lecture 19: Regular rings

Misha Verbitsky

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Associated graded ring (reminder)

DEFINITION: A **multiplicative filtration** on a ring A is a sequence $A = F_0 \supset F_1 \supset ...$ such that all F_i are closed under multiplication and satisfy $F_i F_j \subset F_{i+j}$. A ring equipped with a multiplicative filtration is called **filtered**. An **associated graded quotient** of a filtered ring is $\oplus_{i=0}^{\infty} A^i$, where $A^i = F_i/F_{i+1}$.

CLAIM: $A = F_0 \supset F_1 \supset F_2 \supset ...$ be a filtered ring, $a_1, a_2 \in F_i$ and $b_1, b_2 \in F_j$. Assume that $a_1 = a_2 \mod F_{i+1}$ and $b_1 = b_2 \mod F_{j+1}$. **Then** $a_1 b_1 = a_2 b_2 \mod F_{i+1}F_j + F_i F_{j+1}$.

REMARK: Since $F_{i+1}F_j + F_i F_{j+1} \subset F_{i+j+1}$, the product of $a \in F_i/F_{i+1}$ and $b \in F_j/F_{j+1}$ is well defined as an element of F_{i+j}/F_{i+j+1}. Therefore, **the associated graded ring $\oplus_{i=0}^{\infty} A^i$ is equipped with a natural ring structure.** This ring is called **the associated graded ring**.

EXAMPLE: Let $I \subset A$ be an ideal. Then $A \supset I \supset I^2 \supset ...$ is a multiplicative filtration. The corresponding associated graded ring $A^* := \bigoplus_{i=0}^{\infty} \frac{I^i}{I^{i+1}}$ is called **the associated graded ring of the ideal I.**

Hilbert polynomial of a graded module (reminder)

DEFINITION: Let M^* be a finitely generated graded module over a graded ring A^*, with $A^0 = k$ a field. The **Hilbert function** of M^* is $h_M(n) := \dim_k M^n$.

THEOREM 1: Let M^* be a finitely generated graded module over a graded ring A^* generated by a finite-dimensional space A^1, with $A^0 = k$ a field. **Then the Hilbert function of $h_M(n)$ is polynomial for n sufficiently big.**

THEOREM 2: Let R be a Noetherian local ring, $\mathfrak{m}$ its maximal ideal, $A := \bigoplus_{i=0}^\infty \frac{\mathfrak{m}^i}{\mathfrak{m}^{i+1}}$ its associated graded ring, and $h_A(n) := \dim A^n$. **Then the Hilbert function $h_A(n)$ is a polynomial for $n \gg 0$.** Moreover, **its degree d is equal to the Krull dimension of R.**

COROLLARY: The Krull dimension of a local Noetherian ring R **is equal to the Krull dimension of its associated graded ring.**

Regular rings

THEOREM 3: Let R be a Noetherian local ring of Krull dimension d, $\mathfrak{m}$ its maximal ideal, and $k := R/\mathfrak{m}$ its residue field. **Then** $\dim_k(\mathfrak{m}/\mathfrak{m}^2) \geqslant d$**.** Moreover, **the following are equivalent:**
(i) The associated graded ring of R is isomorphic to $k[t_1, ..., t_d]$;
(ii) $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = d$;
(iii) the ideal $\mathfrak{m}$ can be generated by d elements.

Proof. Step 1: The implication (i) $\Rightarrow$ (ii) is obvious, and (ii) $\Rightarrow$ (iii) follows from Nakayama lemma, which also implies that the associated graded ring A^* of R is generated by A_1. Since A^* is a quotient of $k[A^1]$, and $\dim k[A^1] = \dim A^1$, we obtain $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim k[A^1] \geqslant \dim A^* = \dim R = d$. **It remains to prove (iii) $\Rightarrow$ (i).**

Step 2: Let $\mu : k[A^1] \longrightarrow A^*$ be a natural surjective map, and $I := \ker \mu$. Let $t_1, ..., t_d$ be the basis in A^1; then $k[A^1] = k[t_1, ..., t_d]$. If I is non-empty, there is an algebraic relation between $t_i \in A^1$, which implies $d = \dim \operatorname{tr} A^* < \dim \operatorname{tr} k[A^1] = \dim_k A^1 = d$. **Therefore, $I = 0$ and μ is an isomorphism.** ■

DEFINITION: A Noetherian local ring is called **regular** if (i)-(iii) holds.

Integral closure of a Noetherian ring

DEFINITION: An affine variety $X = \operatorname{Spec}(A)$ is called **smooth** if all local rings of maximal ideals in A are regular.

CLAIM: Discrete valuation rings are regular.

Proof: Indeed, the maximal ideal of the discrete valuation ring R is principal, hence $\mathfrak{m}/\mathfrak{m}^2 = R/\mathfrak{m} = k$. ■

COROLLARY: Let R be a ring of Krull dimension 1 and without zero divisors. **Then R is Dedekind if and only if it is smooth.** ■

Lemma 1: Let R be a Noetherian ring without zero divisors, and $x \in k(R)$ an element in its fraction field. Then **x is integral over R if and only if there is $y \in R$ such that $yx^n \in R$ for all $n > 0$.**

Proof: Let $S = R[x] \subset k(X)$ be the subring generated by x. If $x = a/b$, with $a, b \in R$, we have $b^n S \subset R$, for some $n \in \mathbb{Z}^{>n}$. Then $b^n x^m \in R$ for all $m \in \mathbb{Z}^{\geqslant 0}$.

Conversely, if $yx^n \in R$, then $aS \subset R$; the R-module aS is finitely generated, because R is Noetherian. ■

Regular rings are integrally closed

PROPOSITION: Let R be a Noetherian local ring, $\mathfrak{m}$ its maximal ideal, and $\operatorname{Gr}(R)$ the associated graded ring. Suppose that $\operatorname{Gr}(R)$ contains no zero divisors. **Then R has no zero divisors.** Moreover, **if $\operatorname{Gr}(R)$ is integrally closed then R is also integrally closed**.

Proof. Step 1: Further on, for any $a \in \mathfrak{m}^k \backslash \mathfrak{m}^{k+1}$, **we denote the class of a mod $\mathfrak{m}^{k+1}$ by $\underline{a} \in \operatorname{Gr}(R)$**. Krull's theorem implies that $\coprod \mathfrak{m}^k \backslash \mathfrak{m}^{k+1} = R$, hence $\underline{a} \in \operatorname{Gr}(R)$ is non-zero for any non-zero $a \in R$. If ab vanishes in R, then $\underline{a} \in \mathfrak{m}^k \backslash \mathfrak{m}^{k+1}$ and $\underline{b} \in \mathfrak{m}^l \backslash \mathfrak{m}^{l+1}$ vanishes in $\operatorname{Gr}(R)$, hence $\operatorname{Gr}(R)$ also has zero divisors. **This proves the first statement.**

Step 2: Let $x = a/b$ be integral over R, with $a, b \in R$. Let $R_1 := R/(b)$. By Nakayama lemma, $\bigcap_k \mathfrak{m}^k R_1 = 0$, hence $\bigcap_k (\mathfrak{m}^k + (b)) = (b)$. To show that $a \in (b)$ **it would suffice to prove that $a \in \mathfrak{m}^k + (b)$ for all k.**

Step 3: We prove $a \in \mathfrak{m}^k + (b)$ using induction in k. Assume that $a \in \mathfrak{m}^{k-1} + (b)$, and $a = d + cb$, where $d \in \mathfrak{m}^{k-1} \backslash \mathfrak{m}^k$. For some $y \in R$, and for all $n \in \mathbb{Z}^{>0}$ we have $y \frac{d^n}{b^n} \in R$ (Lemma 1). This brings $yd^n \in b^n R$ and $\underline{y}\underline{d}^n \in \underline{b}^n \operatorname{Gr}(R)$. Since $\operatorname{Gr}(R)$ is integrally closed, this implies that $\frac{\underline{d}}{\underline{b}} \in \operatorname{Gr}(R)$, hence $d = bz \mod \mathfrak{m}^k$, and $a \in \mathfrak{m}^k + (b)$. ■

COROLLARY: Regular rings are integrally closed. ■