Commutative Algebra

lecture 20: Kähler differentials

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Hilbert polynomial of a graded module (reminder)

DEFINITION: Let M^* be a finitely generated graded module over a graded ring A^*, with $A^0 = k$ a field. The **Hilbert function** of M^* is $h_M(n) := \dim_k M^n$.

THEOREM 1: Let M^* be a finitely generated graded module over a graded ring A^* generated by a finite-dimensional space A^1, with $A^0 = k$ a field. **Then the Hilbert function of $h_M(n)$ is polynomial for n sufficiently big.**

THEOREM 2: Let R be a Noetherian local ring, $\mathfrak m$ its maximal ideal, $A := \bigoplus_{i=0}^\infty \frac{\mathfrak m^i}{\mathfrak m^{i+1}}$ its associated graded ring, and $h_A(n) := \dim A^n$. **Then the Hilbert function $h_A(n)$ is a polynomial for $n \gg 0$.** Moreover, **its degree d is equal to the Krull dimension of R.**

COROLLARY: The Krull dimension of a local Noetherian ring R **is equal to the Krull dimension of its associated graded ring.**

Regular rings (reminder)

THEOREM 3: Let R be a Noetherian local ring of Krull dimension d, $\mathfrak{m}$ its maximal ideal, and $k := R/\mathfrak{m}$ its residue field. **Then** $\dim_k(\mathfrak{m}/\mathfrak{m}^2) \geqslant d$**.** Moreover, **the following are equivalent:**
(i) The associated graded ring of R is isomorphic to $k[t_1, ..., t_d]$;
(ii) $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = d$;
(iii) the ideal $\mathfrak{m}$ can be generated by d elements.

Proof. Step 1: The implication (i) $\Rightarrow$ (ii) is obvious, and (ii) $\Rightarrow$ (iii) follows from Nakayama lemma, which also implies that the associated graded ring A^* of R is generated by A_1. Since A^* is a quotient of $k[A^1]$, and $\dim k[A^1] = \dim A^1$, we obtain $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim k[A^1] \geqslant \dim A^* = \dim R = d$. **It remains to prove (iii) $\Rightarrow$ (i).**

Step 2: Let $\mu: \ k[A^1] \longrightarrow A^*$ be a natural surjective map, and $I := \ker \mu$. Let $t_1, ..., t_d$ be the basis in A^1; then $k[A^1] = k[t_1, ..., t_d]$. If I is non-empty, there is an algebraic relation between $t_i \in A^1$, which implies $d = \dim \operatorname{tr} A^* < \dim \operatorname{tr} k[A^1] = \dim_k A^1 = d$. **Therefore, $I = 0$ and μ is an isomorphism.** ■

DEFINITION: A Noetherian local ring is called **regular** if (i)-(iii) holds.

Kähler differentials

DEFINITION: Let R be a ring over a field k, and V an R-module. A k-linear map $D: R \longrightarrow V$ is called **a derivation** if it satisfies **the Leibnitz identity** $D(ab) = aD(b) + bD(a)$. The space of derivations from R to V is denoted $\operatorname{Der}(R, V)$.

REMARK: $\operatorname{Der}(R, V)$ **is an R-module**, with a natural R-action.

DEFINITION: Let R be a ring over a field k. Define an R-module $\Omega^1_k R$ **(the module of Kähler differentials)** with the following generators and relations.

* **Generators** of $\Omega^1_k R$ are indexed by elements of R; for each $a \in R$, the corresponding generator of $\Omega^1_k R$ is denoted da.

* **Relations** in $\Omega^1_k R$ are generated by expressions $d(ab) = adb + bda$, for all $a, b \in R$, and $d\lambda = 0$ for each $\lambda \in k$.

EXERCISE: Prove that the map $d: R \longrightarrow \Omega^1_k R$ mapping a to da **is a derivation**.

Universal property of Kähler differentials

CLAIM: Let V be an R-module, and $D \in \operatorname{Der}(R,V)$ a derivation. Then **there exists a unique R-module homomorphism $\varphi_D : \Omega^1 R \longrightarrow V$ mapping bda to $bD(a)$.**

REMARK: Consider a category $\mathcal{C}$ of R-modules equipped with a derivation $(V, D : R \longrightarrow V)$, and define morphisms in $\mathcal{C}$ as morphisms of R-modules which commute with the derivation map. Then $\Omega^1 R$ **is an initial object in this category.** This is called **the universal property of the module of Kähler differentials**.

CLAIM: $\operatorname{Der}(R,V) = \operatorname{Hom}_R(\Omega^1 R, V)$**.**

Proof: A composition of a derivation $R \xrightarrow{d} \Omega^1 R$ and an R-module homomorphism $\Omega^1 R \longrightarrow V$ lies in $\operatorname{Der}_k(R,V)$. On the other hand, **any derivation $\xi \in \operatorname{Der}_k(R,V)$ is obtained this way,** by the universal property. ■

COROLLARY: $\operatorname{Der}(R) = (\Omega^1 R)^*$**,** where $V^* := \operatorname{Hom}(V,R)$.

EXERCISE: Prove that $\Omega^1(\mathbb{C}[t_1, ..., t_n])$ **is a free $\mathbb{C}[t_1, ..., t_n]$-module generated by $dt_1, ..., dt_n$.**

Kähler differentials for a polynomial ring

CLAIM: Let $R = k[t_1, ..., t_n]$. **Then** $\operatorname{Der}(R)$ **is a free R-module** $\langle d/dt_1, ..., d/dt_n\rangle$ **generated by** $d/dt_1, ..., d/dt_n$**.**

Proof. Step 1: Any derivation is determined by its values on $t_1, ... t_n$. Let $\delta \in \operatorname{Der}(R)$ be a derivation such that $\delta(t_i) = a_i \in R$. Then $\delta(P) = \sum_{i=1}^n a_i \frac{d}{dt_i} P$. **Therefore, the natural map** $\langle d/dt_1, ..., d/dt_n\rangle \to \operatorname{Der}(R)$ **is surjective.**

Step 2: This map is injective, because $\sum_{i=1}^n a_i \frac{d}{dt_i}(t_i) = a_i$, hence **a non-zero expression** $\sum_{i=1}^n a_i \frac{d}{dt_i}$ **produces a non-zero derivation.** ■

PROPOSITION: Let $R = k[t_1, ..., t_n]$. **Then** Ω^1_R **is a free R-module** $\langle dt_1, ..., dt_n\rangle$ **generated by** $dt_1, ..., dt_n$**.**

Proof. Step 1: By Leibnitz rule, Ω^1_R is generated by $dt_1, ..., dt_n$. Let $\Psi : \langle dt_1, ..., dt_n\rangle \to \Omega^1_R$ be the map defined this way. This map is by construction surjective; **it remains to show that Ψ is injective.**

Step 2: For any R-module M, we denote $M^* := \operatorname{Hom}_R(M, R)$ By the universal principle, $(\Omega^1_R)^* = \operatorname{Der}(R)$. Since $\operatorname{Der}(R)$ is a free R-module, $\operatorname{Der}(R)^*$ is a free module with the basis $(ddt_1)^*, ..., (ddt_n)^*$. **The natural map** $R : \Omega^1_R \to (\Omega^1_R)^{**}$ **takes** dt **to** $(ddt_1)^*$**, hence** $\Psi \circ R$ **is injective.** ■

Exact sequence of Kähler differentials

Let $\delta : I \longrightarrow \Omega^1 A$ take x to dx. Since $d(xy) = yd(x) + xd(y)$, $\delta(I^2) = 0$ modulo $I\Omega^1 A$. Therefore, **the map $I/I^2 \xrightarrow{\delta} \Omega^1 A \otimes_A B$ is well defined.**

CLAIM: Let $\varphi : A \longrightarrow B$ be a surjective ring homomorphism, and I its kernel. **Then the following sequence of B-modules is exact: $I/I^2 \xrightarrow{\delta} \Omega^1 A \otimes_A B \xrightarrow{\varphi} \Omega^1 B \longrightarrow 0$.**

Proof: The composition $\delta \circ \varphi$ clearly vanishes, and $\Omega^1 A \otimes_A B \xrightarrow{\varphi} \Omega^1 B$ is clearly surjective. The kernel of $\Omega^1 A \longrightarrow \Omega^1 B$ is generated by $I\Omega^1 A$ together with $d(I)$, hence the kernel of $\Omega^1 A \otimes_A B \xrightarrow{\varphi} \Omega^1 B$ it is equal to $d(I)$. ■

Exact sequence of Kähler differentials (2)

CLAIM 1: Let $\varphi: A \longrightarrow B$ be a surjective ring homomorphism, and I its kernel. Assume that φ admits a section $\sigma: B \longrightarrow A$. **Then the sequence**

$$0 \longrightarrow I/I^2 \xrightarrow{\delta} \Omega^1 A \otimes_A B \xrightarrow{\varphi} \Omega^1 B \longrightarrow 0$$

is exact.

Proof: Consider the map $\rho: A \longrightarrow I/I^2$ taking x to $x - \sigma(\varphi(x))$. Since $(x - \sigma(\varphi(x)))(y - \sigma(\varphi(y))) \in I^2$, we have

$$xy + \sigma(\varphi(x))\sigma(\varphi(y)) = +x\sigma(\varphi(y)) + y\sigma(\varphi(x)).$$

Then

$$\begin{aligned} x(y - \sigma(\varphi(y))) + y(x - \sigma(\varphi(x))) &= 2xy - xy - \sigma(\varphi(x))\sigma(\varphi(y)) \\ &= xy - \sigma(\varphi(x))\sigma(\varphi(y)). \end{aligned}$$

Therefore, ρ is a derivation. Since ρ takes an element of I to itself, **the corresponding universal map $\Omega^1 A \otimes_A B \longrightarrow I/I^2$ is by construction inverse to δ.** This implies that $I/I^2 \xrightarrow{\delta} \Omega^1 A \otimes_A B$ is injective. ■

Zariski cotangent space

DEFINITION: Let $\mathfrak{m} \subset R$ be a maximal ideal, associated with $x \in X = \operatorname{Spec} R$. **The Zariski cotangent space** $T^*_x X$ to $\operatorname{Spec} R$ in $\mathfrak{m}$ is $\frac{\mathfrak{m}}{\mathfrak{m}^2}$.

PROPOSITION: Let R be a ring over k, and $\mathfrak{m} \subset R$ be a maximal ideal which satisfies $\frac{R}{\mathfrak{m}} = k$. Then **the Zariski cotangent space $T^*_{\mathfrak{m}} \operatorname{Spec} R$ is equal to $\Omega^1_k R \otimes_R \frac{R}{\mathfrak{m}}$.**

Proof: Let $R_1 := \frac{R}{\mathfrak{m}}$. Since $\frac{R}{\mathfrak{m}} = k$, the map $R \longrightarrow R_1$ has a section. Then the exact sequence

$$0 \longrightarrow \frac{\mathfrak{m}}{\mathfrak{m}^2} \xrightarrow{\ \delta\ } \Omega^1_k R \otimes_R R^1 \xrightarrow{\ \varphi\ } 0 = \Omega^1_k R_1$$

implies that $\frac{\mathfrak{m}}{\mathfrak{m}^2} = \Omega^1 R \otimes_R R^1$. ∎

COROLLARY: If X is a smooth manifold, $x \in X$, and $\mathfrak{m}_x$ the ideal of all smooth functions vanishing in x. **Then $\mathfrak{m}_x^2$ is the ideal of all functions f such that $f(x) = 0$ and $df|_x = 0$.** In particular, **the natural map $\frac{\mathfrak{m}_x}{\mathfrak{m}_x{}^2} \longrightarrow T^*_x X$ mapping f to $df|_x$ is an isomorphism**.

This explains the term "Zariski cotangent space".

Regular points of an algebraic variety

PROPOSITION: Let $X \subset \mathbb{C}^n$ be an affine variety defined by a system of polynomial equations $f_1(z) = f_2(z) = ... = f_k(z) = 0$, and X_x a localization of $B = \mathcal{O}_B$ in the maximal ideal $\mathfrak{m}_x$ associated with a point $x \in X$. Denote by $W \subset T^*_x\mathbb{C}^n$ the space generated by the differentials $df_1, ..., df_k$. **Then $T^*_x X = \frac{T^*_x\mathbb{C}^n}{W}$.** Moreover, **$B_x$ is regular if and only if $\dim X = n - \dim W$.**

Proof: Let $A = \mathbb{C}[t_1, ..., t_n]$, and I the kernel of the natural surjection $A \longrightarrow B$. Consider the exact sequence $I/I^2 \xrightarrow{\delta} \Omega^1 A \otimes_A B \xrightarrow{\varphi} \Omega^1 B \longrightarrow 0$. Then

$$T^*_x B = \Omega^1 B \otimes_A \frac{A}{\mathfrak{m}_x} = \frac{\Omega^1 A \otimes_A \frac{A}{\mathfrak{m}_x}}{\delta(I/I^2)} = \frac{\langle dt_1, ..., dt_n\rangle}{\langle df_1, .., df_k\rangle} \otimes_A \frac{A}{\mathfrak{m}_x}.$$

Therefore, the Zariski cotangent space $T^*_x X = \Omega^1 B \otimes_A \frac{A}{\mathfrak{m}_x} = \frac{T^*_x\mathbb{C}^n}{W}$ is $(n - \dim_k W)$-dimensional. On the other hand, **B_x is regular if and only if $\dim T^*_x X = \dim X$** (Theorem 3). ■

Regular points and regular values

COROLLARY: Let $X \subset \mathbb{C}^n$ be an affine variety defined by a system of polynomial equations $f_1(z) = f_2(z) = ... = f_k(z) = 0$. Suppose that 0 is a regular value of the map $F(z) = (f_1(z), f_2(z), ..., f_k(z))$. **Then $T_x^* X = \dim X$ for any $x \in X$.** In particular, **all localizations of X in maximal ideals are regular.**

Proof: Since X is given by k equations, dimension of X is at least $n - k$. On the other hand, $\dim T_x^* X = n - \dim W = n - k$ as shown above. By Theorem 3, $\dim T_x^* X \geqslant \dim X$, and **the equality is realized precisely when x is regular.** ■

DEFINITION: A point on an algebraic variety is called **smooth**, if the local ring associated with the corresponding maximal ideal is regular. A variety is called **smooth** if all its points are regular.

Conormal exact sequence

DEFINITION: Let $f: X \to Y$ be a morphism of complex affine varieties associated with a ring homomorphism $f^*: B \to A$. **The module of relative Kähler differentials** $\Omega^1_{X/Y}$ is the module of B-linear Kähler differentials on X, that is, the module generated by da, $a \in A$ are relations $d(uv) = udv + vdu$ $d(b) = 0$ for all $b \in B$.

REMARK: The last condition is equivalent to B-linearity: $d(b) = 0 \Leftrightarrow d(ab) = bd(a)$.

PROPOSITION: (Conormal exact sequence)
Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphism of affine varieties. **Then there is a natural exact sequence**

$$\Omega^1_{Y/Z} \otimes_{\mathcal{O}_Y} \mathcal{O}_X \xrightarrow{\psi} \Omega^1_{X/Z} \xrightarrow{\varphi} \Omega^1_{X/Y} \to 0.$$

Proof. Step 1: Surjectivity in the last term is clear. An image of $\Omega^1_{Y/Z} \otimes_{\mathcal{O}_Y}$ vanishes in $\Omega^1_{X/Y}$ because $dy = 0$ in $\Omega^1_{X/Y}$ for any $y \in \mathcal{O}_Y$. **It remains to prove that this sequence is exact in the middle term.**

Step 2: Let $Q := \Omega^1_{X/Z}/\operatorname{im}\psi$, and $\delta(x)$ be the image of dx in Q for any $x \in \mathcal{O}_X$. This is clearly a $\mathcal{O}_Y$-linear derivation, hence defines an element of $\Omega^1_{X/Y}$. **Therefore,** $\operatorname{im}\Psi \supset \ker\varphi$**.** ■

COROLLARY: Let $X \hookrightarrow Y$ be a variety defined by the ideal I. Then there is a natural exact sequence

$$I/I^2 \to \Omega^1_Y \to \Omega^1_{Y/X} \to 0.$$

Proof: Putting $Z = \operatorname{Spec}(\mathbb{C})$, and writing the conormal exact sequence, we get

$$\Omega^1_Y \otimes_{\mathcal{O}Y} \mathcal{O}_X \to \Omega^1_X \to \Omega^1_{X/Y} \to 0.$$

The surjective map $I/I^2 \to \Omega^1_Y \otimes_{\mathcal{O}Y} \mathcal{O}_X$ was constructed in Claim 1. ■

COROLLARY: Let y be a point on Y. **Then** $\Omega^1_{\{y\}/Y} = T^*_y Y$**.** ■

Étale maps

DEFINITION: Let $f: X\to Y$ be a morphism of complex varieties. We say that f is **unramified** if $\Omega^1_{X/Y}=0$, and **etale** if it is unramified and finite.

REMARK: Caveat emptor: When the base field is not algebraically closed, one needs to ask that f is **flat** as well.

THEOREM: Let $g: Y\to Z$ be an étale map. **Then u^* induces an isomorphism $T^*_{g(y)}Z\to T^*_y Y$** for any $y\in Y$.

Proof: Let $X:=\{y\}$, and $\mathfrak m$ be its maximal ideal. Consider the conormal exact sequence associated with $X\stackrel{f}{\to}Y\stackrel{g}{\to}Z$:

$$\Omega^1_{Y/Z}\otimes_{\mathcal O Y}\left(\frac{\mathcal O_Y}{\mathfrak m}\right)\stackrel{\psi}{\to}\Omega^1_{X/Z}\stackrel{\varphi}{\to}\Omega^1_{X/Y}\to 0.$$

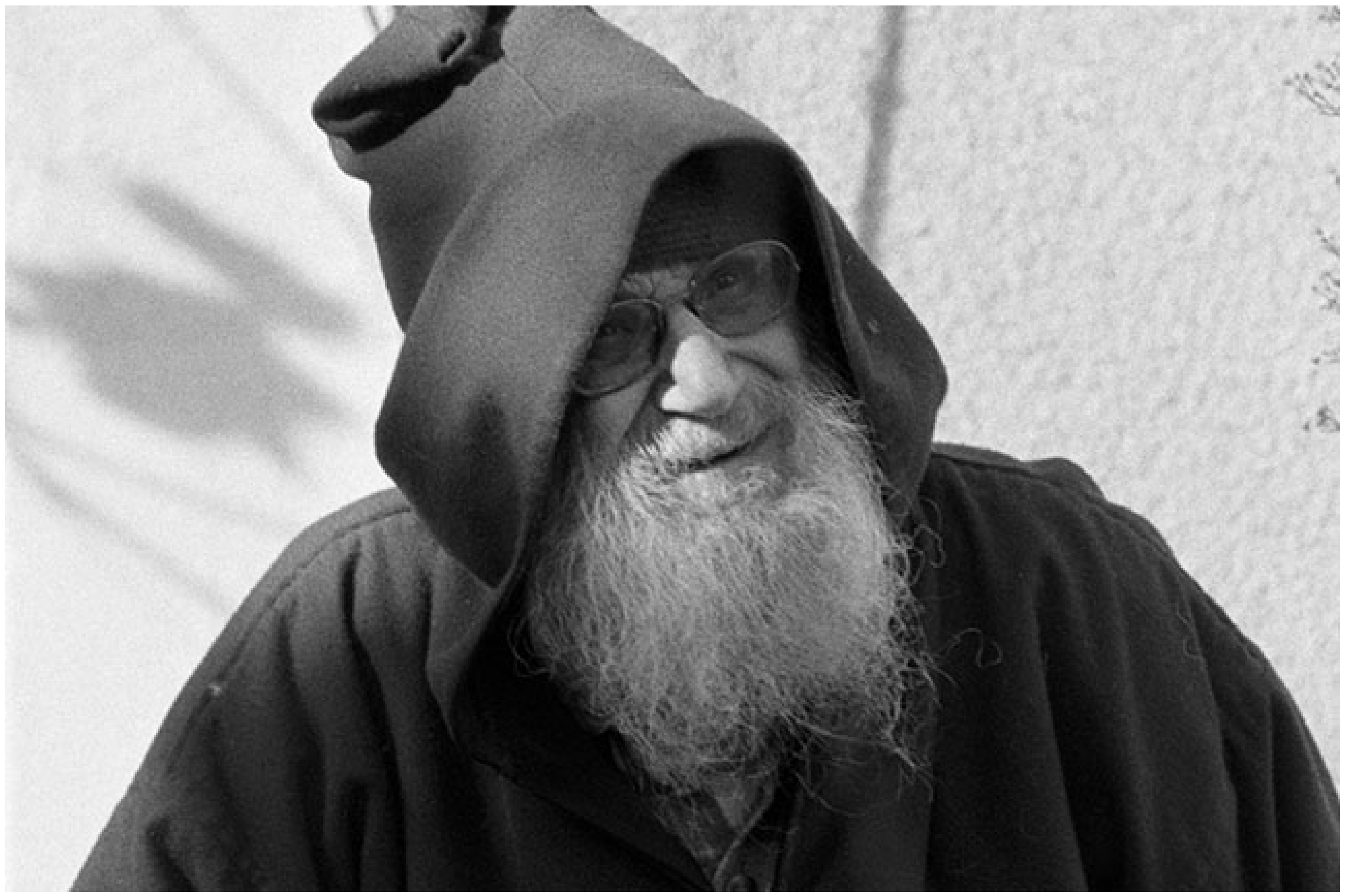
Vanishing of the relative differential implies an isomorphism $\Omega^1_{X/Z}=\Omega^1_{X/Y}$. These spaces are Zariski cotangent spaces to Y and Z because $X=\{x\}$ is a point. ■

COROLLARY: Let $f: X\to Y$ be an étale map. **Then $x\in X$ is smooth if and only if $f(x)$ is smooth.**

Proof: Étale map preserves the Krull dimension and dimension of $\mathfrak m/\mathfrak m^2$. ■

Alexandre Grothendieck

Étale : lisse, sans protubérances (le terme est tiré d'un poème de Victor Hugo sur la mer "étale")



Alexandre Grothendieck (28 March 1928 - 13 November 2014)

Ramification locus

EXERCISE: Ler $[K : k]$ be a finite extension of fields. **Prove that** $\operatorname{Der}_k K = 0$**.**

DEFINITION: Let $X \to Y$ be a finite map. Its **ramification locus** is the set of maximal ideals $\mathfrak{m} \in Y$ such that $\Omega^1_{X/Y}$ localized in $\mathfrak{m}$ is non-zero.

PROPOSITION: The ramification locus is not Zariski dense.

Proof: The intersection of localizations of $\mathcal{O}_Y$ is $\mathcal{O}_Y$. If ramification locus is Zariski dense, we obtain a non-zero differentials (and therefore, non-zero derivations) of the corresponding fraction fields. However, the module $\operatorname{Der}_{k(X)}(k(Y))$ vanishes because $[k(Y) : k(X)]$ is a finite extension. ■

COROLLARY: Any finite map **is etale outside of a Zariski closed set.** ■

COROLLARY: The set of smooth points on an algebraic variety X **is Zariski dense.**

Proof: Noether normalization lemma is used to construct a finite map $X \to \mathbb{C}^n$, and the previous corollary to find smooth points on X. ■