Complex variables 4: Weierstrass division theorem

Rules: This is a class assignment for the next week. Exercises with [*] are extra hard and not necessary to follow the rest. Exercises with [!] are non-trivial, fundamental and necessary for further work.

4.1 Partial fraction decomposition

Definition 4.1. Let f, g be holomorphic functions on a manifold M. **The pole** of the fraction $\frac{f}{g}$ is the set of all points $m \in M$ such that $g = 0$ and the quotient $\frac{f}{g}$ is discontinuous in any neighbourhood of m. **Meromorphic function** is a fraction of holomorphic ones.

Exercise 4.1. Prove that any meromorphic function on $\mathbb{C}^n$ which can be continuously extended to $\mathbb{C}^n$ is holomorphic.

Exercise 4.2. Consider a linear projection $\Pi: \mathbb{C}^n \longrightarrow \mathbb{C}^{n-1}$, and let $\frac{f}{g}$ be a meromorphic function on an open set $U \subset \mathbb{C}^n$ which is holomorphic on the fibers of Π. Prove that $\frac{f}{g}$ is holomorphic.

Definition 4.2. Rational function on $\mathbb{C}$ is a fraction of two polynomials. **Partial fraction** is a rational function $f(z) = \frac{\lambda}{(z-\mu)^k}$, where $k \in \mathbb{Z}^{>0}$ and $\lambda, \mu \in \mathbb{C}$.

Exercise 4.3. (partial fraction decomposition)
Prove that every rational function is a sum of a polynomial and a partial fraction. Prove that such a decomposition is unique.

Exercise 4.4. Prove that statement for rational functions over any algebraically closed field.

Exercise 4.5. Let f be a meromorphic function on a disk without poles on the boundary. Assume that $fdz = dg$, where g is another meromorphic function, and dg denotes the meromorphic differential form $\frac{dg}{dz}dz$. Prove that $\int_{\partial\Delta} fdz = 0$.

Exercise 4.6. Replace g by a smooth function with the same values around $\partial\Delta$ and use the Stokes' formula.

Exercise 4.7. Let a, b be points of the interior of the unit disk $\Delta \subset \mathbb{C}$. Prove that $\int_{\partial\Delta} \frac{1}{(z-b)(z-a)^k} dz = 0$ for any $k \in \mathbb{Z}^{\geqslant 1}$.

Definition 4.3. Recall that L^2**-topology** on the space of functions on a circle S^1 is topology, defined by the norm $|f| = \left(\int_{S^1} |f|^2 dt\right)^{1/2}$

Exercise 4.8. Let f is a continuous complex-valued function on the unit circle $\partial\Delta$.

a. Prove that the function $f_1(a) := \frac{1}{2\pi\sqrt{-1}} \int_{\partial\Delta} \frac{f(z)}{z-a} dz$ is holomorphic in the disk Δ.

b. Prove that $f_1(z)$ is continuously extended to a function $f(z)$ on $\partial\Delta$, or find a counterexample.

c. (*) Prove that the space of functions on S^1 for which f_1 can be continuously extended to f is closed in L^2-topology.

d. (*) Prove that any real function on $\partial\Delta$ can be continuously extended to a harmonic function on Δ, and such extension is unique.

Exercise 4.9. Let f be a meromorphic function on a disk, smoothly extended to its boundary.

a. Prove that $f = f_0 + \sum \frac{b_i}{(z-a_i)^{k_i}}$, where f is holomorphic on the disk, and $|a_i| < 1$ for all i. Prove that such decomposition is unique.

b. Prove the the function $a \mapsto \frac{1}{2\pi\sqrt{-1}} \int_{\partial\Delta} \frac{f(z)}{z-a} dz$ is holomorphic in the interior of Δ.

c. Prove that $f_0(a) = \frac{1}{2\pi\sqrt{-1}} \int_{\partial\Delta} \frac{f(z)}{z-a} dz$.

Hint. Use Exercise 4.7.

4.2 Division with reminders

Exercise 4.10. (Lagrange interpolation polynomial)
Let $z_1, ..., z_n, a_1, ..., a_n$ be complex numbers, with all z_i are pairwise different. Probe that there exists a unique polynomial $P(z)$ of degree $n-1$ such that $P(z_i) = a_i$ for all i.

Exercise 4.11. Let $z_1, ..., z_n, a_1, ..., a_n$ be complex numbers, all z_i are pairwise different, and $k_1, ..., k_n \in \mathbb{Z}^{>0}$. Prove that there exists a unique polynomial $P(z)$ of degree $\sum_{i=1}^n k_i - 1$ such that $P(z) - a_i$ has zero of order k_i in z_i.

Exercise 4.12. (Chinese remainder theorem)
Let $Q_1, ..., Q_n$ be polynomials of degree $\leqslant k_1 - 1, ..., k_n - 1$, with $z_1, ..., z_n$ pairwise different. Prove that there exists a unique polynomial $P(z)$ of degree $\sum_{i=1}^n k_i - 1$ such that $P(z) - Q_i(z)$ has zero of order k_i in z_i.

Exercise 4.13. Let f be a holomorphic function on disk, and g a polynomial of degree k with zeroes in $z_1, ..., z_n$ of order $k_1, ..., k_n$. Prove that there exists a polynomial r of degree $k-1$ such that $f - r$ has zeroes of order $k_1, ..., k_n$ in $z_1, ..., z_n$.

Hint. Use the Chinese remainder theorem.

Exercise 4.14. (division with reminder for a holomorphic function and a polynomial) Let f be a holomorphic function on disk, and g a polynomial of degree k. Prove that there exists a unique holomorphic function h such that $f = gh + r$, where $r(z)$ is a polynomial of degree $< k$.

Hint. Use the previous exercise.

Remark 4.1. Now we shall do the same "division with remainders" operation directly using the Cauchy integral.

Exercise 4.15. (division with remainder using Cauchy formula)
Let $f, g \in \mathbb{C}[z]$ be polynomials. Assume that all zeroes of g belong to interior of the unit disk.

a. Prove that $h(a) := \frac{1}{2\pi\sqrt{-1}} \int_{\partial\Delta} \frac{f(z)}{g(z)} \frac{1}{z-a} dz$ is a polynomial.

b. Prove that $r(z) := f(z) - h(z)g(z)$ has smaller degree than $g(z)$, in other words, $f(z) = h(z)g(z)+r(z)$ is result of division with remainder.

c. Are these statements true for arbitrary polynomial g? Prove or find a counterexample.

Exercise 4.16. Let $f(z)$, $g(z)$ be holomorphic functions on a disk, with g nowhere zero on its boundary. Consider function

$$r(z) := f(z) - g(z) \frac{1}{2\pi\sqrt{-1}} \int_{\partial\Delta} \frac{f(\zeta)}{g(\zeta)} \frac{1}{\zeta - z} d\zeta.$$

a. Prove that $r(z)$ is holomorphic, and

$$r(z) = \frac{1}{2\pi\sqrt{-1}} \int_{\partial\Delta} \frac{f(\zeta)}{g(\zeta)} \frac{g(\zeta) - g(z)}{\zeta - z} d\zeta. \tag{4.1}$$

b. Prove that $f(z) = g(z)h(z) + r(z)$, where $h(z) = \frac{1}{2\pi\sqrt{-1}} \int_{\partial\Delta} \frac{f(\zeta)}{g(\zeta)} \frac{1}{\zeta - z} d\zeta$.

Exercise 4.17 (!). Let g be a polynomial of degree k, and $r(z)$ the function constructed in the previous exercise. Prove that $r(z)$ is a polynomial of degree less than k.

Hint. Use the partial fraction decomposition for $1/g$ and (4.1).

4.3 Weierstrass division theorem

Remark 4.2. As in Weierstrass preparation theorem, we write $(z_1, ..., z_{n-1}, z_n)$ as (z, z_n).

Exercise 4.18. Let $P(z, z_n)$ be a Weierstrass polynomial of degree k, with $P(0, z_n) = z_n^k$.

a. (!) Prove that there exists sufficient small $r, r' > 0$, such that $P(z, z_n)$ is defined in the polydisk $\Delta(n-1, 1) := B_r(z_1, ... z_{n-1}) \times \Delta_{r'}(z_n)$, and $P(z, z_n) \neq 0$ whenever $|z_n| = r'$, $|z| \leqslant r$.

b. (!) Write

$$h(z, z_n) = \frac{1}{2\pi\sqrt{-1}} \int_{\partial\Delta} \frac{F(z, \zeta)}{P(z, \zeta)} \frac{1}{\zeta - z_n} dz.$$

Show that the function $h(z, z_n)$ is holomorphic in $\Delta(n-1, 1)$.

c. Prove that $r := F - Ph$ is a Weierstrass polynomial, holomorphic in $\Delta(n-1, 1)$, and degree less than $\deg P$.

d. Prove uniqueness of the decomposition $F = Ph + r$ with $r(z, z_n)$ Weierstrass polynomial of degree $\leqslant k - 1$.

Exercise 4.19 (!). Consider a holomorphic function $f(z) = \sin(z^2 + w^3)$ on $\mathbb{C}^2$ with coordinates z, w. Find its Weierstrass polynomial.

Exercise 4.20 (**). Prove Weierstrass division theorem for power series: for any $f \in \mathbb{C}[[t_1, ..., t_n]]$ and any polynomial $g \in \mathbb{C}[[t_1, ..., t_{n-1}]][t_n]$ of degree k such that $g(0, 0, ..., 0, t_n) = t_n^k$, there exists a decomposition $f = gh + r$, where $r \in \mathbb{C}[[t_1, ..., t_{n-1}]][t_n]$ and its degree in t_n is less than k.