Complex variables 8: Hilbert Nullstellensatz

Rules: This is a class assignment for the next week. Exercises with [*] are extra hard and not necessary to follow the rest. Exercises with [!] are non-trivial, fundamental and necessary for further work.

8.1 Applications of set theory: Hamel basis in a vector space

Remark 8.1. Feel free to use Zorn lemma, Zermelo theorem and Axiom of Choice. These three statement are equivalent.

Exercise 8.1. Let A, B be sets. Prove that either A is equinumerous to a subset of B, or B is equinumerous to a subset of A.

Exercise 8.2. Prove Kantor-Bernstein-Schroeder theorem: for any two sets A and B, if A is equinumerous to a subset of B, and B is equinumerous to a subset of A, then A and B are equinumerous.

Exercise 8.3 (!). Suppose that there exist surjective maps $A \longrightarrow B$ and $B \longrightarrow A$. Prove that A and B are equinumerous.

Definition 8.1. Let V be a vector space. **Hamel basis** in V is a maximal set of linearly independent vectors of V.

Exercise 8.4. Deduce from Zorn lemma the existence of Hamel basis in any vector space.

Exercise 8.5 (!). Let S be a Hamel basis in W, and $R \subset W$ a collection of vectors generating W. Construct a surjective map $R \times \mathbb{N} \longrightarrow S$.

Hint. Using Zermelo lemma, put a linear ordering on S. For each $r \in R$, write $r = \sum a_i s_i$, $s_i \in S$, with s_i ordered according to the order you have chosen. Prove that the map $(r, i) \mapsto s_i$ is surjective.

Exercise 8.6. Let A be an infinte set. Prove that A can be well ordered in such a way that A has no maximal element.

Exercise 8.7. Let Z be a well ordered set without a maximal element; denote its minimal element by 0.

a. Prove that any $a \in Z$ is contained in an interval $[a, b[$ which is equivalent to $\mathbb{N}$ as an ordered set, and such that the interval $[0, a[$ has no maximal element.

b. Prove that such an interval is unique.

Exercise 8.8 (!). Prove that for any infinite set A, A is equinumerous to $A \times \mathbb{N}$.

Hint. Use the previous exercise.

Exercise 8.9 (!). Let S, S' be two Hamel basises in a vector space V. Prove that they are equinumerous.

Hint. Use all previous exercises starting from 8.3.

8.2 Uncountably-dimensional vector spaces

Definition 8.2. An infinitely-dimensional vector space is called **uncountably-dimensional** if its Hamel basis is uncountable.

Exercise 8.10. Prove that $\mathbb{C}$ is uncountably-dimensional as a vector space over $\mathbb{Q}$.

Exercise 8.11. Prove that $V^* := \operatorname{Hom}_k(V, k)$ is uncountably-dimensional if V is infinitely-dimensional.

Exercise 8.12 (*). Prove that V^* cannot be isomorphic to V if the vector space V is infinitely-dimensional.

Definition 8.3. Let k be a field, and $k(t)$ the field of rational functions over k.

Exercise 8.13. Prove that any set $\frac{1}{t-a_i} \in k(t)$ is linearly independent over k if all $a_i \in k$ are pairwise distinct.

Hint. Take any relation $\sum_{i=1}^n \frac{\lambda_i}{t-a_i}$, multiply by $\prod_{i=1}^n (t-a_i)$, and evaluate in $t = a_1$.

Exercise 8.14 (!). Prove that $\mathbb{C}(t)$ is uncountably-dimensional over $\mathbb{C}$.

Exercise 8.15 (*). Prove that the space of continuois functions on an interval is uncountably-dimensional over $\mathbb{R}$.

Exercise 8.16 (*). Prove that the Hilbert space $L^2(S^1)$ of functions which are square-integrable on a circle is uncountably-dimensional over $\mathbb{R}$.

8.3 Proof of Hilbert Nullstellensatz

Exercise 8.17. Let $K \supsetneq \mathbb{C}$ be a field which strictly contains $\mathbb{C}$. Prove that there exists a $\mathbb{C}$-linear embedding $\mathbb{C}(t) \hookrightarrow K$.

Hint. Prove that this is true for any algebraically closed field in place of $\mathbb{C}$.

Exercise 8.18. Prove that $\mathbb{C}$ has no non-trivial field extensions which are countably-dimensional over $\mathbb{C}$.

Exercise 8.19. Let $I \subset \mathbb{C}[z_1, ..., z_n]$ be a maximal ideal. Prove that either the natural embedding $\mathbb{C} \longrightarrow \mathbb{C}[z_1, ..., z_n]/I$ is an isomorphism, or there exists a $\mathbb{C}$-linear embedding $\mathbb{C}(t) \hookrightarrow \mathbb{C}[z_1, ..., z_n]/I$.

Hint. Use the previous exercise.

Exercise 8.20. Prove that $\mathbb{C}[z_1, ..., z_n]$ is countably-dimensional over $\mathbb{C}$. Deduce that $\mathbb{C}[z_1, ..., z_n]/I$ is countably-dimensional.

Exercise 8.21 (!). Prove that $\mathbb{C}[z_1, ..., z_n]/I = \mathbb{C}$ for any maximal ideal $I \subset \mathbb{C}[z_1, ..., z_n]$.

Exercise 8.22. Let $m \in \mathbb{C}^n$ be a point, I_m the ideal of all functions vanishing in m, and $\phi : \mathbb{C}[z_1, ..., z_n]/I_m \longrightarrow \mathbb{C}$ the natural $\mathbb{C}$-linear isomorphism. Prove that m is a point with coordinates $(\phi(z_1), ..., \phi(z_n))$.

Exercise 8.23 (!). Prove that any maximal ideal $I \subset \mathbb{C}[z_1, ..., z_n]$ is an ideal I_m of all polynomials vanishing in a certain point $m \in \mathbb{C}^n$.

Hint. Use the previous exercise.

Exercise 8.24. Let I be an ideal in $\mathbb{C}[t_1, ..., t_n]$. Denote by $Z_I \subset \mathbb{C}^n$ the set of common zeros of I. Let $F \in \mathbb{C}[t_1, ..., t_n]$ be a polynomial which is non-zero everywhere in Z_I. Prove that

$$1 = aF \mod I$$

for some $a \in \mathbb{C}[t_1, ..., t_n]$.

8.4 Strong Nullstellensatz and localization

Exercise 8.25. ("Rabinowitz trick")
Let $I \subset \mathbb{C}[t_1, ..., t_n]$ be an ideal, Z_I its zero set, and $F \in \mathbb{C}[t_1, ..., t_n]$ a polynomial such that $F = 0$ everywhere on Z_I. Consider an submodule $I' \subset \mathbb{C}[t_1, ..., t_{n+1}]$ generated by all $\psi \in I$ and $\phi := z_{n+1}F - 1$. Prove that $1 \in I'$.

Hint. Prove that I' has no common zeros in $\mathbb{C}^{n+1}$, hence I' is not a proper ideal in $\mathbb{C}[t_1, ..., t_n]$.

Definition 8.4. Localization of a ring R in $F \in R$ is a ring $R[F^{-1}]$, formally generated by elements a/F^n, where $a \in R$, and relations $a/F^n \cdot b/F^m = ab/F^{n+m}$, $a/F^n + b/F^m = \frac{aF^m + bF^n}{F^{n+m}}$ and $aF^k/F^{k+n} = a/F^n$.

Exercise 8.26. Prove that $R[F^{-1}]$ is isomorphic to a quotient $R[t]/(1-tF)$.

Exercise 8.27. Prove that the ring $R[F^{-1}]$ is non-zero if and only if F is not a nilpotent.

Hint. The ring $R[t]/(1-tF)$ is non-zero unless $1 = (1-Ft)P$ for some polynomial $P = \sum_i a_i t^i$. This gives $Fa_{i-1} = a_i$ and $a_0 = 1$.

Exercise 8.28 (!). Construct a bijective correspondence between prime ideals in $A[F^{-1}]$ and prime ideals of A not containing F.

Exercise 8.29 (!). Prove thet the intersection of prime ideals of a ring A is the set of all nilpotent elements of A.

Hint. Use Exercise 8.27.

Exercise 8.30. (”Rabinowitz trick, part 2”)
Let $I \subset \mathbb{C}[t_1, ..., t_n]$ be an ideal, and $F \in \mathbb{C}[t_1, ..., t_n]$ a function which vanishes everywhere on Z_I. Prove that $F^n \in I$.

Hint. Apply Exercise 8.25, to prove that $\mathbb{C}[t_1, ..., t_n][F^{-1}] = 0$. Apply Exercise 8.27 then.

Exercise 8.31. Let $I \subset \mathbb{C}[t_1, ..., t_n]$ be an ideal, Z_I its zero set, and I_{Z_I} the ideal of all functions vanishing in Z_I. Prove that $I = I_{Z_I}$ if and only if I has no nilpotents.

Hint. Use the previous exercise.

Definition 8.5. An algebraic subvariety of $\mathbb{C}^n$ is the set of common zeros of an ideal. A **radical ideal** in a ring is an ideal $I \subset R$ such that R/I has no nilpotents.

Exercise 8.32 (!). (“Strong Nullstellensatz”)
For any ideal $I \subset \mathbb{C}[t_1, ..., t_n]$, let Z_I be its zero set. Prove that this defines a bijective correspondence between radical ideals of $\mathbb{C}[t_1, ..., t_n]$ and algebraic subvarieties of $\mathbb{C}^n$.

Hint. Use the previous exercise.