Complex variables 10: Smooth points and meromorphic maps

Rules: This is a class assignment for the next week. Exercises with [*] are extra hard and not necessary to follow the rest. Exercises with [!] are non-trivial, fundamental and necessary for further work.

10.1 Meromorphic functions

Definition 10.1. Let f, g be holomorphic functions. **Zero divisor** f is the set where $f = 0$. **Pole** of the quotient $\frac{f}{g}$ is the set of all points where $g = 0$ and the fraction $\frac{f}{g}$ is discontinuous. The function $\frac{f}{g}$ is called **a meromorphic function**; it is holomorphic outside of its pole set.

Exercise 10.1. Let $f \in \mathcal{O}_n$, and let $(f) \subset \mathcal{O}_n$ be the principal ideal generated by f. Prove that any ideal $I \supset (f)$ is principal, or find a counterexample.

Hint. Use the Euclidean division in $R[t]$, where R is a factorial ring (Handout 9).

Exercise 10.2. Let $f \in \mathcal{O}_n$. Assume that the prime decomposition of f in the factorial ring $\mathcal{O}_n$ is square-free (in other words, $f = \prod_i f_i$, where f_i are irreducible and distinct). Prove that the ideal (f) is radical.

Exercise 10.3. Let $f, g \in \mathcal{O}_n$. Assume that the zero divisor of f is a subset of the zero divisor of g, and that the prime decomposition of f is square-free. Prove that g is divisible by f.

Hint. Use the previous exercise and apply the Rückert Nullstellensatz.

Exercise 10.4. Let $f, g \in \mathcal{O}_n$ be coprime. Prove that the zero divisor of f is not contained in the zero divisor of g.

Hint. Use the previous exercise.

Exercise 10.5 (!). Prove that the pole Z of a meromorphic function on a smooth complex manifold M is a complex analytic subvariety (that is, Z is locally determined by a system of holomorphic equations).

Hint. Use the previous exercise.

Exercise 10.6 (!). Prove that a bounded meromorphic function on a ball $B \subset \mathbb{C}^n$ is holomorphic.

Hint. Use Exercise 10.4.

Exercise 10.7 (*). Let f be a continuous meromorphic function on a singular complex variety. Prove that f is holomorphic, or find a counterexample.

Definition 10.2. Let $X \subset B^n \subset \mathbb{C}^n$, $Y \subset B^m \subset \mathbb{C}^m$ be complex analytic subsets in open balls, and $\Psi : X \longrightarrow \mathbb{C}^m$ a map defined by m meromorphic functions with poles not containing any open subset of X. If all points if X where Ψ is defined are mapped to Y, we say that Ψ **is a meromorphic map from X to Y**.

Definition 10.3. More generally, a map of complex varieties $\Psi : X \longrightarrow Y$, defined in a dense complement to a proper subvariety $X_1 \subset X$ is called **meromorphic** if locally in a neighbourhood U of each point of X in a chart which identifies U with a complex subvariety of $B \subset \mathbb{C}^n$, the map Ψ is meromorphic as defined above.

Definition 10.4. Complex varieties X, Y are called **bimeromorphic** if there exist meromorphic maps $\psi_1 : X \longrightarrow Y$, $\psi_2 : Y \longrightarrow X$ such that the compositions $\psi_1 \circ \psi_2$ and $\psi_2 \circ \psi_1$ are defined on dense subsets if X and Y and are equal to identity in these sets. In this case the maps ψ_1 is called **a bimeromorphism**, or **a bimeromorphic equivalence**.

Exercise 10.8. Let $\Psi : X \longrightarrow Y$ be a meromorphic map of irreducible complex subvarieties of $\mathbb{C}^n$ or an open ball. Assume that Ψ induces the isomorphism of the fraction fields $\Psi^* : k(\mathcal{O}_Y) \tilde{\longrightarrow} k(\mathcal{O}_X)$. Prove that Ψ is a bimeromorphism. Prove that each bimeromorphic map Ψ induces an isomorphism $\Psi^* : k(\mathcal{O}_Y) \tilde{\longrightarrow} k(\mathcal{O}_X)$.

Definition 10.5. Let $U \subset \mathbb{C}^n$ be an open subset. A complex analytic subset $Z \subset U$ is called **a hypersurface** if it can be defined by a single holomorphic equation $h = 0$, where $h \in \mathcal{O}_U$ is a non-zero holomorphic function.

Exercise 10.9 (!). Prove that each germ of an irreducible complex subvariety $X \subset \mathbb{C}^n$ is bimeromorphic to a germ of a hypersurface $X' \subset \mathbb{C}^m$.

Hint. Use a regular coordinate system, primitive element theorem, and apply the previous exercise.

10.2 Discriminat of a Weierstrass polynomial

Definition 10.6. Let $P(t) = \prod_i (t - \alpha_i)$ be a polynomial. **Discriminant** of P is the product $\prod_{i<j} (\alpha_i - \alpha_j)^2$.

Exercise 10.10. Prove that the discriminant of a polynomial P can be expressed as a polynomial of its coefficients.

Remark 10.1. This allows one to extend the definition of a discriminant to any polynomial $P(t) \in R[t]$, for any ring R.

Exercise 10.11. Express the discriminant of a polynomial $P(t) = t^3 + a_2 t^2 + a_1 t$ as a polynomial of its coefficients.

Exercise 10.12. Prove that the discriminant of a polynomial $P(t) \in \mathcal{O}_n[t]$ is non-zero if and only if $P(t)$ is coprime with its derivative $P'(t)$.

Exercise 10.13. Let Z be a germ of an irreducible complex analytic subvariety in $\mathbb{C}^n$, and $z_1, ..., z_d, z_{d+1}, ..., z_n$ regular coordinates, defined (as usual) in such a way that $J \cap \mathcal{O}_d = 0$ and $J \cap \mathcal{O}_{d+1} \neq 0$, where $J \subset \mathcal{O}_n$ is the ideal of Z. Prove that $[k(\mathcal{O}_Z) : k(\mathcal{O}_d)]$ is a finite field extension, and $z_1, ..., z_n$ generate $k(\mathcal{O}_Z)$ over $k(\mathcal{O}_d)$.

Exercise 10.14. Let Z be a germ of an irreducible complex analytic subvariety in $\mathbb{C}^n$, and $z_1, ..., z_n$ regular coordinates. Prove that a linear function $u = \sum_{i=d+1}^n \lambda_i z_i$ is primitive in $k(\mathcal{O}_Z)$ for general $\lambda_1, ..., \lambda_{d+1} \in \mathbb{C}$.

Hint. Use the previous exercise and apply the argument used to prove the primitive element theorem.

Exercise 10.15 (!). Let Z be a germ of an irreducible complex analytic subvariety, $z_1, ..., z_n$ regular coordinates, $u = \sum_{i=d+1}^n \lambda_i z_i$ a primitive element of $k(\mathcal{O}_Z)$, and $\mathcal{P}_u(t)$ its minimal polynomial, $\mathcal{P}_u(t) \in \mathcal{O}_d[t]$. Denote by $D(\mathcal{P}_u) \in \mathcal{O}_d$ the disciminant of $\mathcal{P}_u(t)$. Prove that $D(\mathcal{P}_u) \neq 0$.

Hint. Use Exercise 10.12.

Exercise 10.16. Let $W(\alpha_1, ..., \alpha_n) = (p_1, ..., p_n)$, where $p_k = \sum_{i=1}^n \alpha_i^k$.

a. Prove that the differential of W is invertible in all points $(\alpha_1, ..., \alpha_n)$ where

$$D(\alpha_1, ..., \alpha_n) := \prod_{i<j} (\alpha_i - \alpha_j) \neq 0.$$

b. Prove that $D(\alpha_1, ..., \alpha_n) = \det(A_{ij})$ where A_{ij} is **the Vandermonde matrix**

$$A_{ij} = \begin{bmatrix} 1 & \alpha_1 & \alpha_1^2 & \dots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \dots & \alpha_2^{n-1} \\ 1 & \alpha_3 & \alpha_3^2 & \dots & \alpha_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_m & \alpha_m^2 & \dots & \alpha_m^{n-1} \end{bmatrix}.$$

c. Prove that the Jacobian (determinant of the differential) of W in the point $\alpha_1, ..., \alpha_n$ is equal to $n! D(\alpha_1, ..., \alpha_n)$.

Exercise 10.17 (!). Let $E(\alpha_1, ..., \alpha_n) = (e_0, ..., e_{n-1})$, where $e_0, ..., e_{n-1}$ are coefficients of a polynomial $P(t) = \prod_i (t - \alpha_i)$. Prove that the differential of E is invertible in all points where $D := \prod_{i<j} (\alpha_i - \alpha_j) \neq 0$.

Hint. Use the previous exercise.

Exercise 10.18 (!). Let $P(t) \in \mathcal{O}_{n-1}(z_n)$ be a Weierstrass polynomial, such that $P(0, ..., 0, z_n) = z_n^k$ and P has zero of multiplicity k in 0, and let $D(P) \in \mathcal{O}_{n-1}$ be its discriminant and $Z \subset \mathbb{C}^n$ the zero set. Denote by $D \subset \mathbb{C}^{n-1}$ the zero set of the discriminat D. Prove that the projection $\Pi : \mathbb{C}^n \longrightarrow \mathbb{C}^{n-1}$ along the last variable induces the covering map $Z \cap \pi^{-1}(U \backslash D) \longrightarrow U \backslash D$ for any sufficiently small neighbourhood $U \subset \mathbb{C}^{n-1}$ of 0.

Hint. Use the previous exercise.

10.3 Smooth points

Definition 10.7. Let $Z \subset \mathbb{C}^n$ be a complex analytic set. A point $z \in Z$ is called **smooth** if in an open neighbourhood $U \subset \mathbb{C}^n$ of z, the intersection $Z \cap U$ is a smooth submanifold of U. Otherwise z is called **singular**. A complex analytic subset $Z \subset \mathbb{C}^n$ is **smooth** if all its points are smooth. **Dimension** of Z in a smooth point $z \in Z$ is dimension of Z as of a smooth complex submanifold of its neighbourhood.

Exercise 10.19. Let $Z \subset \mathbb{C}^n$ be defined by holomorphic equations $f_1, ..., f_k$.

a. Assume that in $z \in Z$ the differentials $df_1, ..., df_k$ are linearly independent. Prove that z is smooth.

b. (*) Suppose that $df_1, ..., df_k$ are linearly dependent in $z \in Z$ and linearly independent in $z' \in Z$. Would it follow that z is a singular point? Prove or find a counterexample.

Exercise 10.20. Let $z \in Z$ be a smooth point in a complex analytic set $Z \subset \mathbb{C}^n$ which has dimension r in z.

a. Prove that in some neighbourhood U of $z \in \mathbb{C}^n$, the set $Z \cap U$ can be given by a system of holomorphic equations $f_1 = ... = f_{n-r} = 0$ in such a way that $df_1, ..., df_{n-r}$ are linearly independent in z.

b. For any collection of defining equations of Z $f_1 = ... = f_{n-r} = 0$, defined in a neighbourhood $U \subset \mathbb{C}^n$, let $Z_{\mathsf{sing}}(df_1, ..., df_{n-r})$ be the set of all $z \in Z$ such that $df_1, ..., df_{n-r}$ are linearly dependent. Prove that $Z_{\mathsf{sing}}(df_1, ..., df_{n-r})$ is complex analytic in U.

c. Prove that the set Z_{sing} of singular points of Z is equal to the intersection of $Z_{\mathsf{sing}}(df_1, ..., df_{n-r})$ taken over all systems of equations $f_1 = ... = f_{n-r} = 0$ defining Z

d. Prove that $Z_{\mathsf{sing}} \subset Z$ is complex analytic.

Exercise 10.21. Find holomorphic functions $f, g \in \mathcal{O}_{\mathbb{C}^n}$ such that df and dg are linearly dependent everywhere on the set $Z := \{x \in \mathbb{C}^n \mid f(z) = g(z) = 0\}$ and linearly independent outside of Z. Can Z be smooth?

Exercise 10.22. Let $P(t) \in \mathcal{O}_{n-1}(z_n)$ be a Weierstrass polynomial such that $P(0, ..., 0, z_n) = z_n^k$ and P has zero of multiplicity k in 0. Let $D(P) \in \mathcal{O}_{n-1}$ be its discriminant and $Z \subset \mathbb{C}^n$ the zero set. Prove that the set of smooth points of Z is open and dense in Z.

Hint. Use Exercise 10.18.

Exercise 10.23 (!). Let Z be a germ of a complex analytic variety. Prove that the set of its smooth points of Z is open and dense in Z.

Hint. Use Exercise 10.9 and the previous exercise.

Exercise 10.24 (*). Let Let $Z \subset \mathbb{C}^n$ be a germ of an irreducible complex analytic hypersurface. Prove that the set of smooth points of Z is connected for some representative of this germ.

Exercise 10.25 (*). Let $Z \subset \mathbb{C}^n$ be an irreducible complex analytic hypersurface. Prove that the set of smooth points of Z is connected.

Exercise 10.26 (*). Let Z be an irreducible complex variety. Prove that the set of smooth points of Z is connected.