Complex variables 11: Divisors and maximum principle

Rules: This is a class assignment for the next week. Exercises with [*] are extra hard and not necessary to follow the rest. Exercises with [!] are non-trivial, fundamental and necessary for further work.

11.1 Divisors

Definition 11.1. Let $Z \subset \mathbb{C}^n$ be a complex analytic subvariety. We call a point $z \in Z$ **smooth** if in a neighbourhood of z, the variety Z is a smooth submanifold of $\mathbb{C}^n$, and **singular** otherwise. A complex analytic subvariety $Z \subset \mathbb{C}^n$ is **smooth** if all its points are smooth. **Dimension** of Z in a smooth point $z \in Z$ is dimension of Z as a complex manifold. **Dimension of** Z is maximum of its dimension in all smooth points.

Definition 11.2. Let $Z, Z_1 \subset \mathbb{C}^n$ be germs of complex varieties in z, z_1, and $\phi: \ Z \longrightarrow Z_1$ map z to z_1. Recall that ϕ is called **a morphism of germs of complex varieties** if it is given by complex analytic functions in a neighbourhood of z. In this situation, the ring $\mathcal{O}_{Z,z}$ of germs of complex analytic functions on Z in z is a $\mathcal{O}_{Z_1,z_1}$-module, with $f \in \mathcal{O}_{Z_1,z_1}$ acting on $\mathcal{O}_{Z,z}$ by $f(a) = \phi^*(f)a$ for any $a \in \mathcal{O}_{Z,z}$. We say that Z **is finite over** Z_1 if $\mathcal{O}_{Z,z}$ is finitely generated as $\mathcal{O}_{Z_1,z_1}$-module, and **dominant** if the image of ϕ does not lie in a proper complex analytic subvariety of Z_1.

Exercise 11.1. Let Z be a complex variety of dimension d. Prove that any connected component of the set $Z \backslash Z_{\mathsf{sing}}$ of smooth points of Z has constant dimension.

Exercise 11.2. Let $Z \subset \mathbb{C}^n$ be an irreducible germ of a complex analytic variety defined by an ideal $J \subset \mathcal{O}_n$. Let d be the maximal number such that $\mathcal{O}_d \cap J = 0$ and $z_1, ..., z_d, ..., z_n$ a regular coordinate system. Prove that Z contains an open, dense subset which is d-dimensional.

Exercise 11.3. Let $\phi: \ Z \longrightarrow Z_1$ be a dominant morphism of germs of complex varieties of $\mathbb{C}^n$.

a. Let $z_1, ..., z_d, ..., z_n$ be regular coordinates on $Z_1 \subset \mathbb{C}^n$. Prove that a non-zero function $f \in \mathcal{O}_d$ does not vanish on Z.[1]

b. (!) Prove that there exists a regular coordinate system $z_1, ..., z_d, ..., z_n$ on Z_1, and a regular coordinate system $x_1, ..., x_{d+k}, ..., x_m$ on Z such that ϕ^* of a coordinate function is a coordinate function.

[1] We interpret the ring $\mathcal{O}_{Z_1} \supset \mathcal{O}_d$ as a subring in $\mathcal{O}_z$ by identifying $f \in \mathcal{O}_{Z_1}$ and $\phi^* f \in \mathcal{O}_z$.

Exercise 11.4 (!). Let $\phi : \ Z \longrightarrow Z_1$ be a finite, dominant morphism. Prove that $\dim Z = \dim Z_1$.

Hint. Use the previous exercise.

Definition 11.3. Let X be a complex variety, $X_1, ..., X_n$ its irreducible components. **Divisor** (Cartier divisor) in a complex variety X is a subvariety $Z \subset X$, not containing any of X_i, and locally given by an equation $f = 0$ for some holomorphic function f.

Exercise 11.5 (!). Let $Z \subset \mathbb{C}^n$ be a germ of a divisor. Prove that any irreducible component of Z is also a divisor.

Exercise 11.6. Let $Z \subset \mathbb{C}^n$ be a divisor. Prove that $\dim Z = n - 1$.

Exercise 11.7 (*). Find a germ of a singular variety X and a Cartier divisor $Y \subset X$ such that the irreducible components of Y are not Cartier divisors.

Exercise 11.8. Let $Y \subset X$ be a divisor which does not belong to the set X_{sing} of singular points of X. Prove that $\dim Y = \dim X - 1$.

Exercise 11.9 (!). Let X_{sing} be the set of singular points of a complex variety X. Prove that $\dim X_{\mathsf{sing}} < \dim X$.

Hint. Construct a finite, dominant morphism from X to $\mathbb{C}^d$, and prove that its differential is invertible outside of a divisor. Then apply Exercise 11.6 and Exercise 11.4.

Exercise 11.10. Let $Y \subset X$ be a divisor, and $\pi : \ X \longrightarrow \mathbb{C}^d$ a finite, dominant morphism. Prove that $\pi(Y)$ is contained in a divisor $D \subset \mathbb{C}^d$.

Hint. Use Exercise 11.18 below.

Exercise 11.11. Let $Y \subset X$ be a divisor. Prove that $\dim Y = \dim X - 1$.

Hint. Use the previous exercise and Exercise 11.6.

Exercise 11.12 (*). Let $Y \subset X$ be a divisor, and X_{sing} the set of singular points of X. Prove that if $Y \subset X_{\mathsf{sing}}$, then Y is a union of irreducible components of X_{sing}.

Exercise 11.13. Prove that any complex variety is not equal to a countable union of its divisors.

Hint. First check it for a smooth manifold.

Exercise 11.14 (!). Prove that a complement to a divisor is always open and dense.

Exercise 11.15. Let $X \subset Y$ be complex varieties. Prove that $\dim X \leqslant Y$.

Exercise 11.16 (*). Let $X = \bigcup X_i$ be an irreducible decomposition of a complex variety, X_{sm} the set of smooth points of X, and f_i a collection of meromorphic functions on each X_i. Define f on X_{sm} writing $f = f_i$ on smooth points of each irreducible component. Prove that for each point $x \in X$, f canbe expressed as a fraction of two holomorphic functions in an appropriate neighbourhood of x.

11.2 Maximum principle

Definition 11.4. **An open map** $\phi : X \longrightarrow Y$ is a continuous map of topological spaces such that $\phi(U)$ is open for any open subset $U \subset X$.

Exercise 11.17. Let f be a non-constant holomorphic function on a smooth, connected complex manifold Z of dimension 1. Prove that f is an open map.

Exercise 11.18. Let $P(z_n) \in \mathcal{O}_{n-1}[z_n]$ be a Weierstrass polynomial with $P(0,0,...,0,z_n) = z_n^k$ and zero of multiplicity k in 0, and $f \in \mathcal{O}_n$ a germ of holomorphic function.

a. Consider a neighbourhood of zero of form $\Delta_r(z_1,...,z_{n-1}) \times \Delta_{r'}(z_n)$, obtained as a product of polydisks of radius r, r'. Prove that for appropriate $r, r' \ll 1$, the projection π to the first $n-1$ coordinates defines a ramified k-sheeted covering $Z \longrightarrow \mathbb{C}^{n-1}$, where Z is the set of zeros of $P(z_n)$.

b. (!) Let $z = z_1,...,z_{n-1}$, and

$$\pi_* f(z) := \frac{1}{2\pi\sqrt{-1}} \int_{\partial\Delta_{r'}} \frac{P'(z_n)}{P(z_n)} f(z, z_n) dz_n.$$

Prove $\pi_* f(z) := \sum_{x \in \pi^{-1}(z)} f(x)$ (counted with multiplicities).

Exercise 11.19. Let $X \xrightarrow{\pi} \mathbb{C}^d$ be a finite morphism of germs of complex varieties, f a holomorphic function on X, and $\pi_* f(z) := \sum_{x \in \pi^{-1}(z)} f(x)$.

a. (!) Prove that $\pi_* f$ is meromorphic.

b. Prove that any meromorphic, bounded function a polydisk is holomorphic.

c. (!) Prove that $\pi_* f$ is always holomorphic.

Hint. To prove (a), produce a meromorphic isomorphism between X and a complex analytic hypersurface $X' \subset \mathbb{C}^{d+1}$, and apply the previous exercise to the meromorphic function on X' which corresonds to f under the isomorphism $k(\mathcal{O}_X) \cong k(\mathcal{O}_{X'})$.

Exercise 11.20 (!). Let f be a holomorphic function on an irreducible complex variety Z of dimension 1. Suppose that $|f|$ has a local non-strict maximum at a point $z \in Z$. Prove that f is constant.

Hint. Use the previous exercise.

Exercise 11.21. Let f be a non-constant holomorphic function on a germ of an irreducible complex variety Z of dimension 1. Prove that the number of solutions of an equation $f(z) = c$ is constant in a neighbourhood of 0.

Hint. Use Exercise 11.18, applied to the map $f: Z \longrightarrow \mathbb{C}$.

Exercise 11.22. Let f be a non-constant holomorphic function on an irreducible complex variety Z of dimension 1. Prove that f is open.

Hint. Use the previous exercise.

Exercise 11.23. Let $Z \subset \mathbb{C}^n$ be a germ of a complex analytic subset, and $x, y \in Z$ two points. Prove that there exists a connected subvariety $Z_1 \subset Z$ of dimension 1 containing x and y.

Exercise 11.24 (!). Let f be a non-constant holomorphic function on an irreducible complex variety Z (any dimension). Prove that f is open.

Hint. Use the previous exercise.

Exercise 11.25. Let f be a non-constant holomorphic function on an irreducible complex variety Z (any dimension). Prove that f is open.

Hint. Use the previous exercise.

Exercise 11.26. Let $Z \subset \mathbb{C}^n$ be a compact complex subvariety. Prove that Z is zero-dimensional.

Hint. Use the previous exercise.