Complex variables 12: Finite morphisms

Rules: This is a class assignment for the next week. Exercises with [*] are extra hard and not necessary to follow the rest. Exercises with [!] are non-trivial, fundamental and necessary for further work.

12.1 Localization (see also Handout 8)

Definition 12.1. Let $S \subset R$ be a subset of a ring R closed under multiplication and not containing 0. **Localization** of R in S, denoted by $R[S^{-1}]$ is a ring formally generated by elements a/F^n, where $a \in R$, and relations $a/F^n \cdot b/G^m = ab/F^nG^m$, $a/F^n + b/G^m = \frac{aG^m + bF^n}{F^nG^m}$ and $aF^k/F^{k+n} = a/F^n$, for all $F, G \in S$, $a, b \in R$.

Definition 12.2. Let R be a ring without zero divisors, and S the set of all non-zero elements of R. **Fraction field** of R is the localization of R in S.

Definition 12.3. A ring A is called **local** if A has only one maximal ideal.

Exercise 12.1. Let $\mathfrak{p}$ be a prime ideal in A. Prove that localization of S in the set of all $F \notin \mathfrak{p}$ is a local ring.

Exercise 12.2 (!). Construct a bijective correspondence between the prime ideals in $A[S^{-1}]$ and the prime ideals of A not containing any $s \in S$.

12.2 Cayley-Hamilton theorem

Exercise 12.3. Let A be a Noetherian ring, and M a finitely generated A-module. Prove that the algebra $\operatorname{End}_A(M)$ of A-linear endomorphisms of M is a finitely generated A-module.

Exercise 12.4. Let A be a Noetherian ring, M a finitely generated A-module, and $\Phi \in \operatorname{End}_A(M)$. Denote by $A[\Phi]$ the subalgebra in $\operatorname{End}_A(M)$ generated by Φ and A. Prove that there exists a monic polynomial $P(t) \in A[t]$ such that $P(\Phi) = 0$.

Hint. Use the previous exercise

Definition 12.4. Let $\Phi \in \operatorname{End}_A(M)$ be an endomorphism of a finitely generated A-module, $e_1, ..., e_n$ a set of generators of M, and $\Phi(e_i) = \sum a_{ij}e_j$. **Characteristic polynomial** $\mathsf{Chpoly}_\Phi(t) \in A[t]$ is determinant of a matrix $\det(t\,\mathsf{Id} - P)$, where $P = (a_{ij})$.

Remark 12.1. Note that $\mathsf{Chpoly}_\Phi(t)$ depends on the choice of generators and on the choice of a presentation $\Phi(e_i) = \sum a_{ij} e_j$.

Exercise 12.5. Find an example of a finitely generated A-module M and $\Phi \in \operatorname{End}_A(M)$ such that $\mathsf{Chpoly}_\Phi(t)$ is not uniquely defined.

Exercise 12.6. Let $M = A^n$ be a free A-module, and $\Phi \in \operatorname{End}_A(M)$. Using the basis $e_1, ..., e_n$ in M, we represent Φ by a matrix $P = (a_{ij})$, where $\Phi(e_i) = \sum a_{ij} e_j$. Denote by $\Lambda^n(M)$ the antisymmetric part of $M^{\otimes n}$.

a. Suppose that A is an algebra over a field $k \supset \mathbb{Q}$. Prove that $\Lambda^n(M)$ is isomorphic to A as A-module.

b. Prove that Φ acts on $\Lambda^n_A(M)$ as $\det \Phi$, where $\det \Phi$ is determinant of a matrix P.

c. (!) Consider the non-degenerate exterior product $\Lambda^{n-1}(M) \times M \xrightarrow{\nu} \Lambda^n(M)$. For any $\rho \in \operatorname{End}_A(M)$, let $\rho\Big|_{\Lambda^{n-1}(M)}$ denote induced action of this endomorphism on $\Lambda^{n-1}(M)$, and $\check{\rho} \in \operatorname{End}_A(M)$ the dual with respect to ν endomorphism of M. Prove that $\rho\check{\rho} = \det(\rho)\, \mathsf{Id}_M$.

Exercise 12.7 (!). Prove the Cayley-Hamilton theorem: $\mathsf{Chpoly}_\Phi(\Phi) = 0$ as endomorphism of M.

Hint. Use the previous exercise to show that $(t - \Phi)(t - \Phi)\check{} = \mathsf{Chpoly}_\Phi(t)$ in $\operatorname{End}_{A[t]}(M \otimes_A A[t])$. Evaluate in $t = \Phi$.

Exercise 12.8 (!). Prove that $\mathsf{Chpoly}_\Phi(\Phi) = 0$ for any endomorphism of a finitely generated A-module.

Hint. Obtain M as a quotient of a free module and reduce the problem to the case when M is free. Then apply the Cayley-Hamilton theorem.

Exercise 12.9. Let M be a finitely generated A-module, $\Phi \in \operatorname{End}_A(M)$, and $I \subset A$ an ideal. Assume that $\Phi(M) \subset IM$. Prove that there is a collection of generators $e_1, ..., e_n$ and a matrix expression $\Phi(e_i) = \sum a_{ij} e_j$ such that $\mathsf{Chpoly}_\Phi(t) = t^n \mod I$.

12.3 Nakayama's lemma

Exercise 12.10. Let $P(t)$ be a characteristic polynomial for an identity endomorphism $\mathsf{Id} \in \operatorname{End}_A(M)$, and $S \in A$ is the sum of all coefficients of P. Prove that $SM = 0$.

Hint. Use the Cayley-Hamilton theorem.

Exercise 12.11 (!). (Nakayama's lemma)
Let M be a finitely generated A-module, and $I \subset A$ an ideal. Assume that $IM = M$. Prove that for some $a \in I$, one has $(1-a)M = 0$.

Hint. Using Exercise 12.9, prove that for an appropriate matrix representation of $\mathsf{Id}_M \in \operatorname{End}_A(M)$, one has $\mathsf{Chpoly}_{\mathsf{Id}_M}(t) = t^n \mod I$. Then apply the previous exercise.

Exercise 12.12 (*). (Krull's theorem)
Let $\mathfrak{a} \subset A$ be an ideal in a Noetherian ring. Prove that $\bigcap \mathfrak{a}^n = 0$.

Definition 12.5. Torsion in an A-module is the kernel of the natural homomorphism $M \longrightarrow M \otimes_A k(A)$. An A-module M is **torsion free** if the map $M \longrightarrow M \otimes_A k(A)$ is injective.

Exercise 12.13. Let $T(M)$ denote the torson submodule of M. Prove that the torsion functor maps an exact sequence $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3$ to an exact sequence

$$0 \longrightarrow T(M_1) \longrightarrow T(M_2) \longrightarrow T(M_3).$$

Exercise 12.14. (Nakayama's lemma for torsion-free modules) Let A be a ring without zero divisors, M a finitely generated torsion-free A-module, and $I \subsetneq A$ an ideal in A, satisfying $IM = M$. Prove that $M = 0$.

Hint. Use Exercise 12.11.

Exercise 12.15 (!). (Nakayama's lemma for local rings)
Let M be a finitely generated module over a Noetherian local ring A. Denote its maximal ideal by $\mathfrak{m}$. Let $M' \subset M$ be its submodule such that $M/\mathfrak{m}M = M'/\mathfrak{m}M'$. Prove that $M = M'$.

Exercise 12.16 (*). Let M be a finitely generated module over a Noetherian ring, and $\phi : M \longrightarrow M$ a surjective homomorphism. Prove that ϕ is an isomorphism.

12.4 Finite morphisms

Definition 12.6. Let $A \xrightarrow{\phi} B$ be a ring homomorphism. Consider B as an A-module. The homomorphism ϕ is called **finite**, or **finite morphism**, or **integral** if B is a finitely generated as an A-module.

Exercise 12.17. Let $A \subset \mathbb{C}[x,y]$ be a subring generated by x^{100}, y^{777} and $xy(x^2-1)$. Prove that the morphism $A \longrightarrow \mathbb{C}[x,y]$ is integral.

Exercise 12.18. Let $\phi: \mathbb{C}[x,y] \longrightarrow \mathbb{C}[w,v]$ be a homomorphism given by $\phi(x) = w^{777}v^{18}$, $\phi(y) = w^{37}v$. Is this morphism finite?

Exercise 12.19 (!). Let $A \subset B$ be a subring, with B finitely generated as an A-module and without zero divisors. Prove that for any maximal ideal $I \subset A$, the ring $B/IB = B \otimes_A (A/IA)$ is finitely generated over A/IA and non-trivial

Hint. To show that B/IB is non-trivial, use the Nakayama's lemma for torsion-free modules.

Exercise 12.20. (Cohen-Seidenberg theorem)
Let $A \subset B$ be a subring, with B finitely generated as an A-module and without zero divisors.

a. (!) Prove that each prime ideal $\mathfrak{p} \subset A$ can be obtained as $\mathfrak{p} = \mathfrak{q} \cap A$, for some prime ideal $\mathfrak{q} \subset B$.

b. (!) Prove that the number of such $\mathfrak{q}$ is finite.

Hint. Use the previous exercise. To pass from a prime ideal $\mathfrak{p} \subset A$ to maximal, localise A and B in all $s \in A \backslash \mathfrak{p}$. To prove finiteness, use the classification of semisimple Artinian rings.

Exercise 12.21. Let $f: X \longrightarrow Y$ be a dominant morphism of germs of complex analytic varieties, such that $\mathcal{O}_X$ is finitely generated over $\mathcal{O}_Y \subset f^*(\mathcal{O}_X)$.

a. (!) Prove that the preimage of any $y \in Y$ is finite and non-empty.

b. Prove that the image of any irreducible complex subvariety $X_1 \subset X$ is an irreducible complex subvariety in Y.

Hint. For the last statement, use the Nullstellensatz.