Complex Variables, 2019: final exam

Rules: Every student receives a list of 9 exercises, and has to solve as many of them as possible by the due date. Please write down the solution and bring it to exam for me to see. To pass the exam you are required to explain the solutions, using your notes. Please learn the proofs of all results you will be using on the way (you may put them in your notes). Please contact me by email `verbit[]impa.br` when you are ready.

The final score N is obtained by summing up the points from the exam problems. Marks: C when $30 \leqslant N < 50$, B when $50 \leqslant N < 70$, A when $70 \leqslant N < 90$, A+ when $N \geqslant 90$.

Lucas Lima receives the even-numbered exercises, Xia Xiao the odd-numbered exercises.

Exercise 1.1 (10 pt). Let X, Y be compact complex varieties, $U \subset X, V \subset Y$ open, dense subsets, and $F: U \longrightarrow V$ a holomorphic map. Prove that F can be extended to a meromorphic map if and only if the closure of the graph of F in $X \times Y$ is complex analytic.

Exercise 1.2 (10 pt). Let $f: X \longrightarrow Y$ be a holomorphic, bijective morphism of germs of complex varieties. Prove that f is invertible, or find a counterexample.

Exercise 1.3 (20 pt). Let X be a germof a complex variety, and R a ring of germs of locally bounded meromorphic functions on X. Prove that every element of R is a root of a monic polynomial over $\mathcal{O}_X$.

Exercise 1.4 (20 pt). Let $f: \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be a holomorphic function. Assume that the differential of f is non-degenerate at some point. Prove that f maps open sets to open sets, or find a counterexample.

Exercise 1.5 (10 pt). Let X be a complex variety, and f a meromorphic function on X which satisfies an equation $P(f) = 0$, where $P(t) \in \mathcal{O}_X[t]$ is a monic polynomial. Prove that f is locally bounded.

Exercise 1.6 (20 pt). A local ring R with maximal ideal $\mathfrak{m}$ is called **strictly Henselian** if its residue field $R/\mathfrak{m}$ is algebraically closed, and every monic polynomial $P \in R[t]$ such that the image of P in $R/\mathfrak{m}[t]$ has no multiple roots, has root in R. Prove that the ring $\mathcal{O}_n$ of germs is strictly Henselian.

Exercise 1.7 (10 pt). Let M be a connected complex variety admitting a non-constant holomophic function. Prove that the ring $\mathcal{O}_M$ of holomorphic functions on M is non-Noetherian.

Exercise 1.8 (10 pt). Let f be a holomorphic function on a smooth complex manifold M, and $V(f)$ its zero set. Prove that each point $z \in V(f)$ has a neighbourhood U such that the intersection $V(f) \cap U$ is connected.

Exercise 1.9 (20 pt). Prove that the set of smooth points in any irreducible complex variety is connected.

Exercise 1.10 (10 pt). Let $Z \subset \mathbb{C}^n$ be a submanifold, given as a zero set of an irreducible homogeneous polynomial. Prove that its germ in 0 is irreducible.

Exercise 1.11 (10 pt). Let f be a non-constant holomorphic function on a connected complex manifold, and $V(f)$ its zero set. Prove that the complement $U \backslash V_f$ is connected, but not simply connected.

Exercise 1.12 (20 pt). Find a simply connected, irreducible complex variety M with a point $m \in M$ such that the fundamental group of the complement $M \backslash \{m\}$ is infinite and non-Abelian.

Exercise 1.13 (20 pt). Let G be a finite group, acting on a complex manifold M holomorphically. Prove that the quotient M/G admits a structure of a complex variety in such a way that the natural map $M \longrightarrow M/G$ is holomorphic.

Exercise 1.14 (10 pt). Let $Z \subset M$ be an irreducible complex analytic subset, and x, y its smooth points. Prove that $\dim_x Z = \dim_y Z$.

Exercise 1.15 (10 pt). Consider a map $\Phi: \mathbb{C} \longrightarrow \mathbb{C}^2$, with $\Phi(z) = (z^2 - z, z^3 - z)$. Prove that the closure of the image of Φ is complex analytic in a neighbourhood of $(0, 0)$.

Exercise 1.16 (10 pt). Let M be a germ in 0 of an irreducible complex variety of dimension $\geqslant 2$, and f a holomorphic function on $M \backslash 0$. Prove that f can be extended to a meromorphic function on M.

Exercise 1.17 (10 pt). Let X_1, X_2 be complex varieties. Consider the space $Z = X_1 \sim X_2$ obtained by the identification of a point $x_1 \in X_1$ with $x_2 \in X_2$. Prove that Z can be equipped with a structure of a complex variety in such a way that the natural maps $X_1 \hookrightarrow Z$ and $X_2 \hookrightarrow Z$ are holomorphic.

Exercise 1.18 (10 pt). Let f be a continuous function on a complex manifold which is holomorphic in an open, dense subset. Prove that f is holomorphic or find a counterexample.