Multiple complex variables

lecture 1: Cauchy theorem

Misha Verbitsky

IMPA, sala 236

March 25, 2019

Complex structure on vector spaces

DEFINITION: Let V be a vector space over $\mathbb{R}$, and $I: V \longrightarrow V$ an automorphism which satisfies $I^2 = -\operatorname{Id}_V$. Such an automorphism is called **a complex structure operator** on V.

We extend the action of I on the tensor spaces $V \otimes V \otimes ... \otimes V \otimes V^* \otimes V^* \otimes ... \otimes V^*$ by multiplicativity: $I(v_1 \otimes ... \otimes w_1 \otimes ... \otimes w_n) = I(v_1) \otimes ... \otimes I(w_1) \otimes ... \otimes I(w_n)$.

Trivial observations:

1. **The eigenvalues α_i of I are $\pm\sqrt{-1}$.** Indeed, $\alpha_i^2 = -1$.

2. **V admits an I-invariant, positive definite scalar product ("metric") g.** Take any metric g_0, and let $g := g_0 + I(g_0)$.

3. **I is orthogonal for such g.** Indeed, $g(Ix, Iy) = g_0(x, y) + g_0(Ix, Iy) = g(x, y)$.

4. **I diagonalizable over $\mathbb{C}$.** Indeed, any orthogonal matrix is diagonalizable.

5. **There are as many $\sqrt{-1}$-eigenvalues as there are $-\sqrt{-1}$-eigenvalues.**

Comples structure operator in coordinates

This implies that in an appropriate basis in $V \otimes_{\mathbb{R}} \mathbb{C}$, the almost complex structure operator is diagonal, as follows:

$$\left[\begin{array}{cccc|cccc} \sqrt{-1} & & & & & & & \\ & \sqrt{-1} & & & & 0 & & \\ & & \ddots & & & & & \\ & & & \sqrt{-1} & & & & \\ \hline & & & & -\sqrt{-1} & & & \\ & 0 & & & & -\sqrt{-1} & & \\ & & & & & & \ddots & \\ & & & & & & & -\sqrt{-1} \end{array}\right]$$

We also obtain its normal form in a real basis:

$$\begin{bmatrix} 0 & -1 & & & & & & \\ 1 & 0 & & & & & & \\ & & 0 & -1 & & & & \\ & & 1 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & \ddots & & \\ & & & & & & 0 & -1 \\ & & & & & & 1 & 0 \end{bmatrix}$$

The Grassmann algebra

DEFINITION: Let V be a vector space. Denote by $\Lambda^i V$ the space of antisymmetric polylinear i-forms on V^*, and let $\Lambda^* V := \bigoplus \Lambda^i V$. Denote by $T^{\otimes i} V$ the algebra of **all** polylinear i-forms on V^* ("tensor algebra"), and let $\mathrm{Alt} : T^{\otimes i} V \longrightarrow \Lambda^i V$ be **the antisymmetrization**,

$$\mathrm{Alt}(\eta)(x_1, ..., x_i) := \frac{1}{i!} \sum_{\sigma \in \Sigma_i} (-1)^{\tilde{\sigma}} \eta(x_{\sigma_1}, ..., x_{\sigma_i})$$

where Σ_i is the group of permutations, and $\tilde{\sigma} = 1$ for odd permutations, and 0 for even. Consider the multiplicative operation ("wedge-product") on $\Lambda^* V$, denoted by $\eta \wedge \nu := \mathrm{Alt}(\eta \otimes \nu)$. The space $\Lambda^* V$ with this operation is called **the Grassmann algebra**.

REMARK: It is an algebra of anti-commutative polynomials.

Properties of Grassmann algebra:

1. $\dim \Lambda^i V := \binom{\dim V}{i}$, $\dim \Lambda^* V = 2^{\dim V}$.

2. $\Lambda^*(V \oplus W) = \Lambda^*(V) \otimes \Lambda^*(W)$.

The Hodge decomposition in linear algebra

DEFINITION: Let (V, I) be a space equipped with a complex structure. **The Hodge decomposition** $V \otimes_{\mathbb{R}} \mathbb{C} := V^{1,0} \oplus V^{0,1}$ is defined in such a way that $V^{1,0}$ is a $\sqrt{-1}$-eigenspace of I, and $V^{0,1}$ a $-\sqrt{-1}$-eigenspace.

REMARK: Let $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$. The Grassmann algebra of skew-symmetric forms $\Lambda^n V_{\mathbb{C}} := \Lambda^n_{\mathbb{R}} V \otimes_{\mathbb{R}} C$ admits a decomposition

$$\Lambda^n V_{\mathbb{C}} = \bigoplus_{p+q=n} \Lambda^p V^{1,0} \otimes \Lambda^q V^{0,1}$$

We denote $\Lambda^p V^{1,0} \otimes \Lambda^q V^{0,1}$ by $\Lambda^{p,q} V$. The resulting decomposition $\Lambda^n V_{\mathbb{C}} = \bigoplus_{p+q=n} \Lambda^{p,q} V$ is called **the Hodge decomposition of the Grassmann algebra**.

De Rham algebra

DEFINITION: Let M be a smooth manifold. **A bundle of differential i-forms on M** is the bundle $\Lambda^i T^* M$ of antisymmetric i-forms on TM. It is denoted $\Lambda^i M$.

REMARK: $\Lambda^0 M = C^\infty M$.

DEFINITION: Let $\bigotimes_k T^* M \xrightarrow{\Pi} \Lambda^k M$ be the antisymmetrization map,

$$\Pi(\alpha)(x_1, ..., x_n) := \frac{1}{n!} \sum_{\sigma \in \mathrm{Sym}_n} (-1)^\sigma \alpha(x_{\sigma_1}, x_{\sigma_2}, ..., x_{\sigma_n}).$$

Define **the exterior multiplication** $\wedge : \Lambda^i M \times \Lambda^j M \longrightarrow \Lambda^{i+j} M$ as $\alpha \wedge \beta := \Pi(\alpha \otimes \beta)$, where $\alpha \otimes \beta$ is a section $\Lambda^i M \otimes \Lambda^j M \subset \bigotimes_{i+j} T^* M$ obtained as their tensor multiplication.

REMARK: The fiber of the bundle $\Lambda^* M$ at $x \in M$ **is identified with the Grassmann algebra $\Lambda^* T_x^* M$.** This identification is compatible with the Grassmann product.

De Rham differential

THEOREM: There exists a unique operator $C^\infty M \xrightarrow{d} \Lambda^1 M \xrightarrow{d} \Lambda^2 M \xrightarrow{d} \Lambda^3 M \xrightarrow{d} ...$ satisfying the following properties

1. On functions, d is equal to the differential.
2. $d^2 = 0$
3. $d(\eta\wedge\xi) = d(\eta)\wedge\xi + (-1)^{\tilde\eta}\eta\wedge d(\xi)$, where $\tilde\eta = 0$ where $\eta\in\lambda^{2i}M$ is **an even form,** and $\eta\in\lambda^{2i+1}M$ is **odd.**

DEFINITION: The operator d is called **de Rham differential**.

DEFINITION: A form η is called **closed** if $d\eta = 0$, **exact** if $\eta\in \operatorname{im} d$. The group $\frac{\ker d}{\operatorname{im} d}$ is called **de Rham cohomology** of M.

Holomorphic functions

DEFINITION: Let $U \subset \mathbb{C}^n$ be an open subset, and $f: U \longrightarrow \mathbb{C}$ a function of class C^1 (differentiable at least once). We say that f is **holomorphic** if the differential $df: T_xU \longrightarrow \mathbb{C}$ is complex linear at each $x \in U$.

REMARK: Clearly, f is holomorphic if and only if $df \in \Lambda^{1,0}(U)$, where $\Lambda^{1,0}(U)$ is the Hodge (1,0)-component of the de Rham algebra.

Taylor series decomposition for holomorphic functions in 1 variable is implied by the Cauchy formula: for any folomorphic function f in disk $\Delta \subset \mathbb{C}$,

$$\int_{\partial\Delta} \frac{f(z)dz}{z-a} = 2\pi\sqrt{-1}\, f(a),$$

where $a \in \Delta$ any point, and z coordinate on $\mathbb{C}$. Indeed, in this case,

$$2\pi\sqrt{-1}\, f(a) = \sum_{i\geqslant 0} a^i \int_{\partial\Delta} f(z)(z^{-1})^{i+1},$$

because $\frac{1}{z-a} = z^{-1} \sum_{i\geqslant 0} (az^{-1})^i$.

Cauchy formula in dimension 1

Let's prove Cauchy formula, using Stokes' theorem. Since the space $\Lambda^{1,0}\mathbb{C}$ is 1-dimensional, $df \wedge dz = 0$ for any holomorphic function on $\mathbb{C}$. This gives

CLAIM: A function on a disk $\Delta \subset \mathbb{C}$ **is holomorphic if and only if the form $\eta := f dz$ is closed** (that is, satisfies $d\eta = 0$). ■

Now, let S_ε be a radius ε circle around a point $a \in \Delta$, Δ_ε its interior, and $\Delta_0 := \Delta \backslash \Delta_\varepsilon$. Stokes' theorem gives

$$0 = \int_{\Delta_0} d\left(\frac{f(z)dz}{z-a}\right) = -\int_{S_\varepsilon} \frac{f(z)dz}{z-a} + \int_{\partial\Delta} \frac{f(z)dz}{z-a},$$

hence Cauchy formula would follow if we show that $\lim\limits_{\varepsilon\to 0} \int_{S_\varepsilon} \frac{f(z)dz}{z-a} = 2\pi\sqrt{-1}f(a)$.

Assuming for simplicity $a=0$ and parametrizing the circle S_ε by $\varepsilon e^{\sqrt{-1}\,t}$, we obtain

$$\int_{S_\varepsilon} \frac{f(z)dz}{z} = \int_0^{2\pi} \frac{f(\varepsilon e^{\sqrt{-1}\,t})}{\varepsilon e^{\sqrt{-1}\,t}} d(\varepsilon e^{\sqrt{-1}\,t}) =$$

$$= \int_0^{2\pi} \frac{f(\varepsilon e^{\sqrt{-1}\,t})}{\varepsilon e^{\sqrt{-1}\,t}} \sqrt{-1}\,\varepsilon e^{\sqrt{-1}\,t} dt = \int_0^{2\pi} f(\varepsilon e^{\sqrt{-1}\,t})\sqrt{-1}\,dt$$

as ε tends to 0, $f(\varepsilon e^{\sqrt{-1}\,t})$ tends to $f(0)$, and this integral goes to $2\pi\sqrt{-1}f(0)$.

Holomorphic functions on $\mathbb{C}^n$

THEOREM: Let $f: U \longrightarrow \mathbb{C}$ be a differentiable function on an open subset $U \subset \mathbb{C}^n$. **Then the following are equivalent.**
(1) f **is holomorphic**.
(2) For any complex affine line $L \in \mathbb{C}^n$, the restriction $f|_L = \mathbb{C}$ is **holomorphic as a function of one complex variable.**
(3) f is expressed as a sum of Taylor series around any point $(z_1, ..., z_n) \in U$: for all sufficiently small $t_1, ..., t_n$, one has $f(z_1 + t_1, z_2 + t_2, ..., z_n + t_n) = \sum_{i_1,...,i_n} a_{i_1,...,i_n} t_1^{i_1} t_2^{i_2} ... t_n^{i_n}$.

Proof: Equivalence of (1) and (2) is clear, because a restriction of $\theta \in \Lambda^{1,0}(M)$ to a line is a $(1,0)$-form on a line, and, conversely, if df is of type (1,0) on each complex line, it is of type (1,0) on TM, which is implied by the following linear-algebraic observation.

LEMMA: Let $\eta \in V^* \otimes \mathbb{C}$ be a complex-valued linear form on a real vector space (V, I) equipped with a complex structure I. **Then $\eta \in \Lambda^{1,0}(V)$ if and only if its restriction to any I-invariant 2-dimensional subspace L belongs to $\Lambda^{1,0}(L)$.**
EXERCISE: Prove it.

(3) clearly implies (1). (1) implies (3) by Cauchy formula (many variables), proven below.

Cauchy formula (many variables)

REMARK: Let $U \subset \mathbb{C}^n$ be an open subset, and $z_1, ..., z_n$ complex coordinates. Holomorphicity of $f: U \longrightarrow \mathbb{C}$ is equivalent to $df \in \Lambda^{1,0}(M)$, which is equivalent to $df \wedge dz_1 \wedge dz_1 \wedge ... \wedge dz_n = 0$. Denote the form $dz_1 \wedge dz_1 \wedge ... \wedge dz_n$ by Φ. We obtain that f **is holomorphic if and only if the form** $f\Phi$ **is closed**

THEOREM: (Cauchy formula in dimension n)
Let $\Delta \subset \mathbb{C}^n$ be a polydisk (product of disks) of radius 1, and $\alpha_1, ..., \alpha_n \in \Delta$ complex numbers. Denote by $S \subset \mathbb{C}^n$ the product of circles of radius 1 in variables $z_1, ..., z_n$:, $S = S_1(z_1) \times S_1(z_2) \times ... \times S_1(z_n)$. Let f be a holomorphic function in a polydisk. **Then** $\int_S V = (2\pi\sqrt{-1}\,)^n f(\alpha_1, ... \alpha_n)$**, where**

$$V = \frac{f\Phi}{(z_1 - \alpha_1)(z_2 - \alpha_2) \times ... \times (z_n - \alpha_n)}.$$

Proof. Step 1: Denote by Z the set $\bigcup_{i=1}^n \{(z_1, ..., z_n) \mid z_i = \alpha_i\}$. The complement of Z is the set of definition of the closed differential form V. Let S_ε be the product of circles of radius ε with center in $\alpha_1, ..., \alpha_n$. Then $S, S_\varepsilon \subset \mathbb{C}^n \backslash Z$, and **the tori S, S_ε are homotopy equivalent in the domain $\mathbb{C}^n \backslash Z$, where V is closed. It remains to show that** $\lim_{\varepsilon \to 0} \int_{S_\varepsilon} V = (2\pi\sqrt{-1}\,)^n f(\alpha_1, ... \alpha_n)$**.**

Cauchy formula (many variables), part 2

THEOREM: (Cauchy formula in dimension n)
Let $\Delta \subset \mathbb{C}^n$ be a polydisk (product of disks) of radius 1, and $\alpha_1, ..., \alpha_n \in \Delta$ complex numbers. Denote by $S \subset \mathbb{C}^n$ the product of circles of radius 1 in variables $z_1, ..., z_n$:, $S = S_1(z_1) \times S_1(z_2) \times ... \times S_1(z_n)$. Let f be a holomorphic function in a polydisk. **Then** $\int_S V = (2\pi\sqrt{-1}\,)^n f(\alpha_1, ...\alpha_n)$, **where**

$$V = \frac{f\Phi}{(z_1-\alpha_1)(z_2-\alpha_2)...(z_n-\alpha_n)}.$$

Proof. Step 1: Let S_ε be a product of circles of radius ε with center in $\alpha_1, ..., \alpha_n$. **It remains to show that** $\lim_{\varepsilon\to 0}\int_{S_\varepsilon} V = (2\pi\sqrt{-1}\,)^n f(\alpha_1, ...\alpha_n)$.

Step 2: To simplify notation we set $\alpha_i = 0$. Parametrize S_ε by the cube $[0, 2\pi]^n$ using the map $t_1, ..., t_n \longrightarrow \varepsilon e^{\sqrt{-1}\,t_1}, ..., \varepsilon e^{\sqrt{-1}\,t_n}$. This gives

$$\int_{S_\varepsilon} V = \int_{S_\varepsilon} f(z)\frac{dz_1}{z_1}\wedge ... \wedge \frac{dz_n}{z_n} =$$

$$= \int_0^{2\pi} ... \int_0^{2\pi} \frac{f(\varepsilon e^{\sqrt{-1}\,t_1}, \varepsilon e^{\sqrt{-1}\,t_2}, ..., \varepsilon e^{\sqrt{-1}\,t_n})}{\varepsilon e^{\sqrt{-1}\,t_1}\varepsilon e^{\sqrt{-1}\,t_2}...\varepsilon e^{\sqrt{-1}\,t_n}}\varepsilon^n d\left(e^{\sqrt{-1}\,t_1}\right) d\left(e^{\sqrt{-1}\,t_2}\right)...d\left(e^{\sqrt{-1}\,t_n}\right) =$$

$$= (\sqrt{-1}\,)^n \int_0^{2\pi} ... \int_0^{2\pi} f(\varepsilon e^{\sqrt{-1}\,t_1}, ..., \varepsilon e^{\sqrt{-1}\,t_n})dt_1 dt_2...dt_n,$$

which converges to $(2\pi\sqrt{-1}\,)^n f(0, ..., 0)$. ■

Cauchy formula and Taylor expansion

REMARK: Cauchy formula implies that **holomorphic functions defined in a polydisk have Taylor expansion in this polydisk**. Indeed,

$$f(\alpha_1, ...\alpha_n) = \frac{1}{(2\pi\sqrt{-1}\,)^n} \int_S \frac{f dz_1 \wedge ... \wedge dz_n}{(z_1 - \alpha_1)(z_2 - \alpha_2) \times ... \times (z_n - \alpha_n)}$$

Take the Taylor expansion of $(z_i - \alpha_i)^{-1}$ using

$$\frac{1}{(z_i - \alpha_i)} = \frac{z_i^{-1}}{(1 - \alpha_i z_i^{-1})} = \sum_{l=0}^{\infty} \alpha_i^l z_i^{-l-1}.$$

Then

$$f(\alpha_1, ...\alpha_n) = \sum_{i_1=0}^{\infty} ... \sum_{i_n=0}^{\infty} \alpha_1^{i_1} .... \alpha_{i_n}^{i_n} \int_{S_\varepsilon} f(z_1, ..., z_n) z_1^{-i_1-1} ... z_n^{-i_n-1} dz_1 \wedge ... \wedge dz_n.$$