Class test 1: Grassmann algebra

Exercise 1.1. Let $\omega \in \Lambda^2 V$ be a non-degenerate anti-symmetric 2-form on a vector space. Prove that the multiplication by ω defines an injective map $\Lambda^1(V) \longrightarrow \Lambda^3(V)$.

Exercise 1.2. Let $\omega \in \Lambda^2(V)$, $V = \mathbb{R}^{2n}$ be a non-degenerate 2-form. Prove that multiplication by ω^{n-1} induces an isomorphism $\Lambda^1(V) \xrightarrow{\wedge\omega^{n-1}} \Lambda^{2n-1}(V)$.

Exercise 1.3. Let $V = \mathbb{R}^4$, and $\alpha \in \Lambda^2 V$ be a non-zero antisymmetric 2-tensor. Prove that there exists $\beta \in \Lambda^2 V$ such that $\alpha \wedge \beta \neq 0$.

Exercise 1.4. Let $\omega_1, \omega_2 \in \Lambda^2(\mathbb{R}^4)$ symplectic forms satisfying $\omega_1 \wedge \omega_2 = 0$. Prove that $(\omega_1 + \omega_2) \wedge (\omega_2 - \omega_1) = 0$ or find a counterexample.