Complex analytic spaces 1: Hodge decomposition

Rules: This is a class assignment for the next week. Exercises with [*] are extra hard and not necessary to follow the rest. Exercises with [!] are non-trivial, fundamental and necessary for further work.

1.1 Complex structures on real vector spaces

Definition 1.1. Let V be a vector space over a field k. A linear operator $A \in \operatorname{End}_k(V)$ is **semisimple** if it can be diagonalized over the algebraic closure $\bar{k}$. It is **of finite order** if $A^n = \mathsf{Id}_V$ for sone integer $n > 0$.

Exercise 1.1. Prove that any operator of finite order over a field k of characteristic 0 is semisimple

Exercise 1.2 (*). Find a non-semisimple operator of finite order over a field k of characteristic p.

Definition 1.2. Complex structure on a real vector space V is an endomorphism $I \in \operatorname{End}(V)$ satisfying $I^2 = -\mathsf{Id}_V$.

From now on, V is considered a real vector space.

Exercise 1.3. Let $F: V \longrightarrow V$ be an operator which satisfies $F^3 = -F$. Prove that F is semisimple. Prove that F is written in some (real) basis by a matrix

$$\begin{pmatrix} \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & & & & & & \\ & \ddots & & & & & \\ & & \ddots & & & & \\ & & & \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} & & & \\ & & & & \ddots & & \\ & & & & & \ddots & \\ & & & & & & \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \end{pmatrix}$$

Exercise 1.4. Let V be a real vector space equipped with an action of a group $G = \mathbb{Z}/n\mathbb{Z}$, $n > 2$, such that non-unit elements of G act on V by linear automorphisms without non-zero fixed vectors. Prove that V admits a G-invariant complex structure operator.

Exercise 1.5 (*). Prove that the space of all complex structures $I: V \longrightarrow V$ is homotopy equivalent to the space of all non-degenerate skew-symmetric 2-forms $\omega \in \Lambda^2 V$.

Exercise 1.6. Let I be a complex structure on a vector space V of dimension $2n$. Prove that the group of invertible matrices $A \in GL_{\mathbb{R}}(V)$ satisfying $A \circ I = I \circ A$ is isomorphic to $GL(n, \mathbb{C})$.

1.2 Algebras defined by generators and relations

Definition 1.3. Let V be a vector space. **Free** or **tensor** algebra generated by V is an algebra $T(V) := \bigoplus_i V^{\otimes i}$ with multiplivation given by $x \cdot y = x \otimes y$. The zero component $V^{\otimes 0}$ is identified with the ground field. Therefore, $T(V)$ is an algebra with unit.

Exercise 1.7. Let V be a vector space over the ground field k, called "the space of generators", and $W \subset T(V)$ another space called "the space of relations". Consider the quotient space $A := \frac{T(V)}{T(V)WT(V)}$, where $T(V)WT(V)$ is a subspace of $T(V)$ generated by vectors vwv', where $w \in W$, $v, v' \in T(V)$. Assume that A is non-zero. Prove that A is equipped with a natural structure of an algebra with unit, in such a way that the quotient map $T(V) \longrightarrow A$ is an homomorphism.

Definition 1.4. In assumptions of the previous exercise, let v_i be a basis in V, and w_i basis in W. Each relation $w_i = 0$ can be written as a non-commutative tensorial expression

$$\sum_I \alpha_{i_1,\ldots,i_n} x_{i_1} x_{i_2} \cdots = 0$$

where I runs through a set of multi-indices $i_1, \ldots, i_n$, for various n, and $\alpha_{i_1,\ldots,i_n} \in k$ are scalar coefficients. The algebra A is called **algebra with generators** v_i **and relations** $w_i = 0$.

Definition 1.5. Let V be a 3-dimensional space over $\mathbb{R}$, with basis I, J, K, and $\mathbb{H}$ an algebra generated by V with relations $I^2 = J^2 = K^2 = I \cdot J \cdot K = -1$. Then $\mathbb{H}$ is called **quaternion algebra**.

Exercise 1.8. Prove that quaternion algebra is a 4-dimensional algebra with division.

Hint. Use the same argument which was used to show that the complex numbers have division.

Exercise 1.9. a. Prove that any algebra A with unit can be defined by generators and relations.

b. (*) Prove that when A is finite-dimensional, this can be done in such a way that the space of generators V and the space of relations W are finitely-dimensional.

Definition 1.6. An algebra A defined by the space of generators V and the space of relations W is called **finitely generated** if V can be chosen finitely-dimensional, and **finitely represented** if both W and V can be chosen finite-dimensional.

Exercise 1.10. a. Prove that the matrix algebra $\operatorname{Mat}(\mathbb{R}^2)$ is finitely represented.

b. Prove that the algebra $k[t, t^{-1}]$ of Laurent polynomials is finitely represented.

Exercise 1.11 (*). Find a finitely generated algebra which is not finitely represented.

Definition 1.7. Let V be a vector space with a bilinear symmetric form $g: V \otimes V \longrightarrow \mathbb{R}$. Consider the algebra $\mathrm{Cl}(V)$ generated by V, with relations

$$v_1 \cdot v_2 + v_2 \cdot v_1 = g(v_1, v_2) \cdot 1,$$

for all $v_1, v_2 \in V$. This algebra is called **Clifford algebra** over k.

Exercise 1.12. Describe all Clifford algebras over $\mathbb{R}$ for $\dim V = 1, 2$.

Exercise 1.13. Prove that the following algebras are isomorphic to a Clifford algebra over $\mathbb{R}$ for an appropriate space V with bilinear symmetric form g, and find this V and g.

a. $\mathbb{C}$

b. (!) $\mathbb{H}$

c. (!) $\mathrm{Mat}(2, \mathbb{R})$

d. (*) $\mathrm{Mat}(4, \mathbb{R})$.

Exercise 1.14 (*). Let $n = \dim V$. Find $\dim \mathrm{Cl}(V)$.

1.3 Grassmann algebra

Definition 1.8. An algebra A is called **graded** if A is represented as $A = \bigoplus A^i$, where $i \in \mathbb{Z}$, and the product satisfies $A^i \cdot A^j \subset A^{i+j}$. Instead $\bigoplus A^i$ one often writes A^*, where $*$ denotes all indices together. Some of the spaces A^i can be zero, but the ground field is always assumed to belong in A^0.

Example: The tensor algebra $T(V)$ and the polynomial algebra are obviously graded.

Definition 1.9. A subspace $W \subset A^*$ of a graded algebra $A^* = \bigoplus_i A^i$ is called **graded** if W is a direct sum of components $W^i \subset A^i$.

Exercise 1.15. Let $W \subset T(V)$ be a graded subspace. Prove that the algebra generated by V with relation space W is also graded.

Definition 1.10. Let V be a vector space, and $W \subset V \otimes V$ a graded subspace, generated by vectors $x \otimes y + y \otimes x$ and $x \otimes x$, for all $x, y \in V$. A graded algebra defined by the generator space V and the relation space W is called **Grassmann algebra**, or **exterior algebra**, and denoted $\Lambda^*(V)$. The space $\Lambda^i(V)$ is called ***i*-th exterior power** of V, and the multiplication in $\Lambda^*(V)$ – **exterior multiplication**. Exterior multiplication is denoted $\wedge$.

Remark 1.1. Grassmann algebra is a Clifford algebra with the symmetric form $g = 0$.

Exercise 1.16. Prove that $\Lambda^1 V$ is isomorphic to V.

Exercise 1.17. Let V be finitely dimensional. Prove that $\Lambda^2(V)^*$ is isomorphic to the space of bilinear skew-symmetric forms on V.

Exercise 1.18. Consider a subalgebra $\bigoplus_{i\in\mathbb{Z}} \Lambda^{2i}(V)$ in a Grassmann algebra. Prove that this subalgebra is commutative.

Definition 1.11. An element of Grassmann algebra is called **even** if it lies in $\bigoplus_{i\in\mathbb{Z}} \Lambda^{2i}(V)$ and **odd** if it lies in $\bigoplus_{i\in\mathbb{Z}} \Lambda^{2i+1}(V)$. For an even or odd $x \in \Lambda^*(V)$, we define a number $\tilde{x}$ called **parity** of x. The parity of x is 1 for even x and -1 for odd.

Exercise 1.19 (!). Prove that $x \wedge y = (-1)^{\tilde{x}\tilde{y}} y \wedge x$.

Exercise 1.20 (*). Find all $\eta \in \Lambda^2(V)$ such that $\eta^2 = 0$.

Exercise 1.21. Let $x_1, x_2, \ldots$ be a basis in $V \cong \Lambda^1 V$. Show that the set of vectors $x_{i_1} \wedge x_{i_2} \wedge x_{i_3} \wedge \cdots$, for all $i_1 < i_2 < i_3 < \ldots$ is a basis in $\Lambda^*(V)$.

Exercise 1.22 (!). Let V be a d-dimensional vector space. Find $\dim \Lambda^i(V)$. Prove that $\dim \Lambda^d V = 1$.

Definition 1.12. The space $\Lambda^d V$ is called **the space of determinant vectors on** V.

Exercise 1.23. Let V be a d-dimensional vector space, $x_1, x_2, \ldots, x_d$ its basis, and $\det := x_1 \wedge x_2 \wedge x_3 \cdots \wedge x_d$ the corresponding determinant vector in $\Lambda^d V$. For a given permutation $I = (i_1, i_2, \ldots, i_d)$ consider a vector $I(\det) := x_{i_1} \wedge x_{i_2} \wedge x_{i_3} \cdots \wedge x_{i_d}$. Prove that $I(\det) = \pm \det$. Prove that this correspondence gives a homomorphism σ from the group S_d of permutations to $\{\pm 1\}$. Prove that this homomorphism maps a product of odd number of transpositions to -1 and a product of even number of transpositions to 1.

Definition 1.13. The number $\sigma(I)$ is called **signature** of a permitation I.

Definition 1.14. Let $\eta \in V^{\otimes d}$ be an element in the d-th tensor power of V. The group S_d acts on $V^{\otimes d}$ by permutation of tensor factors. Define $\mathsf{Alt}(\eta)$ as

$$\mathsf{Alt}(\eta) := \frac{1}{d!} \sum_{I \in S_d} \sigma(I) I(\eta).$$

This operation is called **antisymmetrization**. We say that a vector $\eta \in V^{\otimes d}$ is **totally antisymmetric** if $\eta = \mathsf{Alt}(\eta)$.

Exercise 1.24. Let $\eta \in V^{\otimes d}$ be a vector which satisfies $\eta = \frac{1}{d!} \sum_{I \in S_d} I(\eta)$. Prove that $I(\eta) = \eta$ for any permutation $I \in S_d$.

Exercise 1.25 (!). Let $\eta \in V^{\otimes d}$ be a totally antisymmetric tensor. Prove that $I(\eta) = \sigma(I)\eta$ for any permutation $I \in S_d$.

Exercise 1.26. Prove that $\mathsf{Alt}(\mathsf{Alt}(\eta)) = \mathsf{Alt}(\eta)$ for any $\eta \in V^{\otimes d}$.

Exercise 1.27. Let $W \subset V \otimes V$ be the space of relations of Grassmann algebra defined above. Prove that $\mathsf{Alt}(T(V) \cdot W \cdot T(V)) = 0$.

Remark 1.2. From this exercise it follows that there exists a natural map from $\Lambda^i(V)$ to the space $\mathsf{im}\,\mathsf{Alt}$ of totally antisymmetric tensors.

Exercise 1.28 (!). Prove that the homomorphism $\Lambda^i(V) \longrightarrow \mathsf{im}\,\mathsf{Alt}$ defined above is bijective.

Exercise 1.29 (!). In the previous exercise, we have identified $\Lambda^*(V)$ and the space of totally antisymmetric tensors. This defines multiplicative structure on the space of totally antisymmetric tensors. Prove that this multiplicative structure can be written as follows. Given totally antisymmetric tensors $\alpha, \beta \in T(V)$, to find $\alpha \wedge \beta \in \mathsf{im}\,\mathsf{Alt} = \Lambda^*(V)$, we muptiply α and β in $T(V)$ and apply Alt.

Remark 1.3. From now on, we identify $\Lambda^*(V)$ and the space of totally antisymmetric tensors, and consider $\Lambda^*(V)$ as a subspace in the tensor algebra.

Exercise 1.30. Let V_1, V_2 be vector spaces. Prove that $\Lambda^*(V_1 \oplus V_2)$ and $\Lambda^*(V_1) \otimes \Lambda^*(V_2)$ are isomorphic as graded vector spaces.

Exercise 1.31. Prove that $\dim \Lambda^*(V) = 2^{\dim V}$.

Exercise 1.32. Consider the map

$$V \otimes \Lambda^i(V) \xrightarrow{\wedge} \Lambda^{i+1}(V),$$

defined by $x \otimes \eta \mapsto x \wedge \eta$. For any given η, this defines a linear map $L_\eta : V \longrightarrow \Lambda^{i+1}(V)$.

a. (*) Prove that for all $\eta \neq 0$ one has $\dim \ker L_\eta \leqslant i$.

b. (*) Suppose that $\dim \ker L_\eta = i$. Prove that in this case $\eta = x_1 \wedge x_2 \wedge \cdots \wedge x_i$ for some $x_1, \ldots, x_i \in V$.

1.4 Determinant

Exercise 1.33. Let W be a one-dimensional vector space over k. Prove that $\operatorname{End} W$ is naturally isomorphic to k.

Exercise 1.34. Let $A \in \operatorname{End}(V)$ be a linear endomorphism of a vector space V. Prove that the action of A on $V \cong \Lambda^1 V$ is uniquely extended to a multiplicative endomorphism of the algebra $\Lambda^* V$. Prove that this homomorphism preserves the grading.

Definition 1.15. Let V be a d-dimensional vector space and $A \in \operatorname{End}(V)$. Consider the induced endomorphism of the space of determinant vectors $\Lambda^d(V)$ denoted as $\det A \in \operatorname{End}(\Lambda^d(V))$. Since $\Lambda^d(V)$ is 1-dimensional, the space $\operatorname{End}(\Lambda^d(V))$ is naturally identified with k. This allows to consider $\det A$ as a number, that is, an element of k. This number is called **determinant** of A.

Exercise 1.35. Let V be a vector space, and $x_1, \ldots, x_d \in V$. Prove that $x_1 \wedge x_2 \wedge \cdots \wedge x_d \neq 0$ if and only if these vectors are linearly independent.

Exercise 1.36 (!). Prove that $A \in \operatorname{End}(V)$ has positive-dimensional kernel if and only if $\det A = 0$.

Hint. Use the previous exercise.

Exercise 1.37 (!). Prove that det defines a homomorphism from the group $GL(V)$ of invertible matrices to the multiplicative group k^* of the ground field.

Exercise 1.38 (!). Let V, V' be vector spaces, A, A' their endomorphisms. Then $A \oplus A'$ defines an endomorphism of $V \oplus V'$. Prove that $\det(A \oplus A') = \det A \det A'$.

Hint. Use the isomorphism $\Lambda^*(V \oplus V') \cong \Lambda^*(V) \otimes \Lambda^*(V')$.

Exercise 1.39 (*). Let V be a vector space equipped with a non-degenerate bilinear form, that is, an isomorphism $g: \ V \longrightarrow V^*$, and A a linear operator preserving g. Prove that $\det A = \pm 1$.

Exercise 1.40 (!). Let $V = \mathbb{R}^n$, and $\alpha \in \Lambda^m V$, $\alpha \neq 0$. Prove that there exists $\beta \in \Lambda^{n-m} V$ such that $\alpha \wedge \beta \neq 0$.

1.5 Hodge decomposition

Exercise 1.41. Let (V, I) be a real vector space equipped with a complex structure operator. Prove that the corresponding complex vector space $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$ is decomposed as $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$, with $I\big|_{V^{1,0}} = \sqrt{-1}$ and $I\big|_{V^{1,0}} = -\sqrt{-1}$. Prove that $\dim_{\mathbb{C}} V^{1,0} = \dim_{\mathbb{C}} V^{0,1} = \frac{1}{2} \dim_{\mathbb{R}} V$.

Exercise 1.42 (!). In these assumptions, let $\Lambda^{p,0}(V) := \Lambda^p_{\mathbb{C}}(V^{1,0})$, $\Lambda^{0,q}(V) := \Lambda^p_{\mathbb{C}}(V^{0,q})$, and $\Lambda^{p,q}(V) := \Lambda^{p,0}(V) \otimes_{\mathbb{C}} \Lambda^{0,q}(V)$. Prove that $\Lambda^*(V) = \bigoplus_{p,q} \Lambda^{p,q}(V)$.

Definition 1.16. The decomposition $\Lambda^*(V) = \bigoplus_{p,q} \Lambda^{p,q}(V)$ is called **the Hodge decomposition** on the Grassmann algebra.

Exercise 1.43 (!). Let $\dim_{\mathbb{R}} V = 2n$, and $\Phi \in \Lambda^{n,0}(V)$ be a non-zero element. Consider $x \in V_{\mathbb{C}}$. Prove that $x \in V^{1,0}$ if and only if $x \wedge \Phi = 0$.

Exercise 1.44 (*). Let $\eta \in \Lambda^2 V$ be a non-degenerate 2-form, and $L_\eta: \ \Lambda^k V \longrightarrow \Lambda^{k+2} V$ the multiplication map $x \longrightarrow x \wedge \eta$. Prove that $L_\eta^k: \ \Lambda^1 V \longrightarrow \Lambda^{1+2k} V$ is injective if $\dim V > 2k$.