Complex variables 2: Sheaves and manifolds

Rules: This is a class assignment for the next week. Exercises with [*] are extra hard and not necessary to follow the rest. Exercises with [!] are non-trivial, fundamental and necessary for further work.

2.1 Smooth manifolds

Definition 2.1. A **cover** of a topological space X is a family of open sets $\{U_i\}$ such that $\bigcup_i U_i = X$. A cover $\{V_i\}$ is a **refinement** of a cover $\{U_i\}$ if every V_i is contained in some U_i.

Exercise 2.1. Show that any two covers of a topological space admit a common refinement.

Definition 2.2. A cover $\{U_i\}$ is an **atlas** if for every U_i, we have a map $\varphi_i : U_i \to \mathbb{R}^n$ giving a homeomorphism of U_i with an open subset in $\mathbb{R}^n$. The **transition maps**

$$\Phi_{ij} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

are induced by the above homeomorphisms. An atlas is **smooth** if all transition maps are smooth (of class C^∞, i.e., infinitely differentiable), **smooth of class** C^i if all transition functions are of differentiability class C^i, and **real analytic** if all transition maps admit a Taylor expansion at each point.

Definition 2.3. A **refinement** of an **atlas** is a refinement of the corresponding cover $V_i \subset U_i$ equipped with the maps $\varphi_i : V_i \to \mathbb{R}^n$ that are the restrictions of $\varphi_i : U_i \to \mathbb{R}^n$. Two atlases (U_i, φ_i) and (U_i, ψ_i) of class C^∞ or C^i (with the same cover) are **equivalent** in this class if, for all i, the map $\psi_i \circ \varphi_i^{-1}$ defined on the corresponding open subset in $\mathbb{R}^n$ belongs to the mentioned class. Two arbitrary atlases are **equivalent** if the corresponding covers possess a common refinement giving equivalent atlases.

Definition 2.4. A **smooth structure** on a manifold (of class C^∞ or C^i) is an atlas of class C^∞ or C^i considered up to the above equivalence. A **smooth manifold** is a topological manifold equipped with a smooth structure.

Remark 2.1. This is a terrible definition, but it is given in (almost) all textbooks.

Exercise 2.2 (*). Construct an example of two nonequivalent smooth structures on $\mathbb{R}^n$.

Exercise 2.3 (**). Prove that the cardinality of the set of smooth structures on $\mathbb{R}^n$ is no more than continuum.

Definition 2.5. A **smooth function** on a manifold M is a function f whose restriction to the chart (U_i, φ_i) gives a smooth function $f \circ \varphi_i^{-1} : \varphi_i(U_i) \longrightarrow \mathbb{R}$ for each open subset $\varphi_i(U_i) \subset \mathbb{R}^n$.

Remark 2.2. It is easier to define manifolds using sheaves.

Definition 2.6. A **presheaf of functions** on a topological space M is a collection of subrings $\mathcal{F}(U) \subset C(U)$ in the ring $C(U)$ of all functions on U, for each open subset $U \subset M$, such that the restriction of every $\gamma \in \mathcal{F}(U)$ to an open subset $U_1 \subset U$ belongs to $\mathcal{F}(U_1)$.

Definition 2.7. A presheaf of functions $\mathcal{F}$ is called **a sheaf of functions** if these subrings satisfy the following condition. Let $\{U_i\}$ be a cover of an open subset $U \subset M$ (possibly infinite) and $f_i \in \mathcal{F}(U_i)$ a family of functions defined on the open sets of the cover and compatible on the pairwise intersections:

$$f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$$

for every pair of members of the cover. Then there exists $f \in \mathcal{F}(U)$ such that f_i is the restriction of f to U_i for all i.

Remark 2.3. A presheaf of functions is a collection of subrings of functions on open subsets, compatible with restrictions. **A sheaf of fuctions** is a presheaf allowing "gluing" a function on a bigger open set if its restriction to smaller open sets lies in the presheaf.

Definition 2.8. A sequence $A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow ...$ of homomorphisms of abelian groups or vector spaces is called **exact** if the image of each map is the kernel of the next one.

Exercise 2.4. Let $\mathcal{F}$ be a presheaf of functions. Show that $\mathcal{F}$ is a sheaf if and only if for every cover $\{U_i\}$ of an open subset $U \subset M$, the sequence of restriction maps

$$0 \to \mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \to \prod_{i \neq j} \mathcal{F}(U_i \cap U_j)$$

is exact, with $\eta \in \mathcal{F}(U_i)$ mapped to $\eta\Big|_{U_i \cap U_j}$ and $-\eta\Big|_{U_j \cap U_i}$.

Exercise 2.5. Show that the following spaces of functions on $\mathbb{R}^n$ define sheaves of functions.

a. Space of continuous functions.

b. Space of smooth functions.

c. Space of functions of differentiability class C^i.

d. (*) Space of functions which are pointwise limits of sequences of continuous functions.

e. Space of functions vanishing outside a set of measure 0.

Exercise 2.6. Show that the following spaces of functions on $\mathbb{R}^n$ are presheaves, but not sheaves

a. Space of constant functions.

b. Space of bounded functions.

c. Space of functions vanishing outside of a bounded set.

d. Space of continuous functions with finite $\int |f|$.

Definition 2.9. A **ringed space** $(M, \mathcal{F})$ is a topological space equipped with a sheaf of functions. A **morphism** $(M, \mathcal{F}) \stackrel{\Psi}{\longrightarrow} (N, \mathcal{F}')$ of ringed spaces is a continuous map $M \stackrel{\Psi}{\longrightarrow} N$ such that, for every open subset $U \subset N$ and every function $f \in \mathcal{F}'(U)$, the function $f \circ \Psi$ belongs to the ring $\mathcal{F}\big(\Psi^{-1}(U)\big)$. An **isomorphism** of ringed spaces is a homeomorphism Ψ such that Ψ and Ψ^{-1} are morphisms of ringed spaces.

Remark 2.4. Usually the term "ringed space" stands for a more general concept, where the "sheaf of functions" is an abstract "sheaf of rings," not necessarily a subsheaf in the sheaf of all functions on M. The above definition is simpler, but not standard.

Exercise 2.7. Let M, N be open subsets in $\mathbb{R}^n$ and let $\Psi : M \to N$ be a smooth map. Show that Ψ defines a morphism of spaces ringed by smooth functions.

Exercise 2.8. Let M be a smooth manifold of some class and let $\mathcal{F}$ be the space of functions of this class. Show that $\mathcal{F}$ is a sheaf.

Exercise 2.9 (!). Let M be a topological manifold, and let (U_i, φ_i) and (V_j, ψ_j) be smooth structures on M. Show that these structures are equivalent if and only if the corresponding sheaves of smooth functions coincide.

Remark 2.5. This exercise implies that the following definition is equivalent to the one stated earlier.

Definition 2.10. Let $(M, \mathcal{F})$ be a topological manifold equipped with a sheaf of functions. It is said to be a **smooth manifold** of **class** C^∞ or C^i if every point in $(M, \mathcal{F})$ has an open neighborhood isomorphic to the ringed space $(\mathbb{R}^n, \mathcal{F}')$, where $\mathcal{F}'$ is a ring of functions on $\mathbb{R}^n$ of this class.

Definition 2.11. A **coordinate system** on an open subset U of a manifold $(M, \mathcal{F})$ is an isomorphism between $(U, \mathcal{F})$ and an open subset in $(\mathbb{R}^n, \mathcal{F}')$, where $\mathcal{F}'$ are functions of the same class on $\mathbb{R}^n$.

Remark 2.6. In order to avoid complicated notation, from now on we assume that all manifolds are Hausdorff and smooth (of class C^∞). The case of other differentiability classes can be considered in the same manner.

Exercise 2.10 (!). Let $(M, \mathcal{F})$ and $(N, \mathcal{F}')$ be manifolds and let $\Psi : M \to N$ be a continuous map. Show that the following conditions are equivalent.
(i) In local coordinates, Ψ is given by a smooth map
(ii) Ψ is a morphism of ringed spaces.

Remark 2.7. An isomorphism of smooth manifolds is called a **diffeomorphism**. As follows from this exercise, a diffeomorphism is a homeomorphism that maps smooth functions onto smooth ones.

Exercise 2.11 (*). Let $\mathcal{F}$ be a presheaf of functions on $\mathbb{R}^n$. Figure out a minimal sheaf that contains $\mathcal{F}$ in the following cases.

a. Constant functions.

b. Functions vanishing outside a bounded subset.

c. Bounded functions.

Exercise 2.12 (*). Describe all morphisms of ringed spaces from $(\mathbb{R}^n, C^{i+1})$ to $(\mathbb{R}^n, C^i)$.

2.2 Complex manifolds

Definition 2.12. Let $U \subset \mathbb{C}^n$ be an open subset, with $z_i = x_i + \sqrt{-1}y_i$ the standard coordinate system. **Standard almost complex structure operator** is a map $I: TU \longrightarrow TU$ such that $I(d/dx_i) = d/dy_i$, $I(d/dy_i) = -d/dx_i$. We extend I to the cotangent bundle in the usual way. Using the eigenvalue decomposition for I, we define **the Hodge decomposition** $\Lambda^1(U, \mathbb{C}) = \Lambda^{1,0}(U) \oplus \Lambda^{0,1}(U)$, with I acting on $\Lambda^{1,0}(U)$ as $\sqrt{-1}$ and on $\Lambda^{0,1}(U)$ as $\sqrt{-1}$. A function $f: U \longrightarrow \mathbb{C}$ is called **holomorphic** if $df \in \Lambda^{1,0}(U)$.

Remark 2.8. Let $U \subset \mathbb{C}^n$ be an open subset, and $\mathcal{O}_U$ the ring of holomorphic functions. Clearly, $\mathcal{O}_U$ is a sheaf of rings of (complex-valued) functions on U.

Definition 2.13. A complex manifold is a ringed space which is locally isomorphic to $(U, \mathcal{O}_U)$, where $U \subset \mathbb{C}^n$ is an open subset and $\mathcal{O}_U$ denotes the sheaf of holomorphic functions on U

Definition 2.14. Coordinate system on an open subset $U \subset M$ of a complex manifold is an isomorphism between U, considered as a ringed space, and $(B, \mathcal{O}_B)$, where $B \subset \mathbb{C}^n$ is an open subset. The coordinates $z_1, ..., z_n$ on B are called **coordinate functions**.

Exercise 2.13. Let U, V be open subsets on a complex manifold equipped with coordinate systems $\phi: U \longrightarrow \mathbb{C}^n$, $\psi: V \longrightarrow \mathbb{C}^n$. Consider a map of open subsets of $\mathbb{C}^n$ ("the gluing map"), considered as a map from an open subset of $\mathbb{C}^n$ to another open subset of $\mathbb{C}^n$, $\psi\phi^{-1}: \phi(U \cap V) \longrightarrow \psi(U \cap V)$. Prove that it is holomorphic.

Exercise 2.14 (*). Let $f_1, ..., f_n$ be holomorphic functions on a complex manifold m, $\dim_{\mathbb{C}} M = n$. Assume that the differentials $df_1, ..., df_n$ are linearly independent in $x \in M$. Prove that there exists a coordinate system $U \ni x$ such that $f_1, ..., f_n$ are coordinate functions.

Exercise 2.15 (!). Let $(M, \mathcal{O}_M)$ be a connected complex manifold, and f, g two non-zero holomorphic functions. prove that $fg \neq 0$.

2.3 Almost complex manifolds.

Definition 2.15. Almost complex structure on a smooth manifold M is an operator $I \in \operatorname{End} TM$ satisfying $I^2 = -\operatorname{Id}_{TM}$. Then (M, I) is called **an almost complex manifold**. **Hodge decomposition** on a cotangent bundle to an almost complex manifold is the decomposition $\Lambda^1(M) \otimes_{\mathbb{R}} \mathbb{C} = \Lambda^{1,0}(M) \oplus \Lambda^{0,1}(M)$, where $I\Big|_{\Lambda^{1,0}(M)} = \sqrt{-1}$, and $I\Big|_{\Lambda^{0,1}(M)} = -\sqrt{-1}$. Function $f: M \longrightarrow \mathbb{C}$ on an almost complex manifold is called **holomorphic** if $df \in \Lambda^{1,0}(M)$.

Remark 2.9. An almost complex structure on any open subset in $\mathbb{C}^n$ was given in Definition 2.12.

Exercise 2.16. Let f be a function on $\mathbb{C}^n$ which restricts to any line $\mathbb{C} \subset \mathbb{C}^n$ holomorphically. Prove that f is holomorphic.

Definition 2.16. A subset $X \subset M$ is called **a complex analytic subset** if it is locally obtained as a set of common zeroes of a collection of holomorphic functions (locally defined). In other words, for any $x \in X$ there exists an open neighbourhood $U \subset M$ containing x and a collection of holomorphic functions such that X is the set of common zeroes of this collection.

Exercise 2.17. Let $f: \mathbb{C} \longrightarrow \mathbb{C}$ be a smooth function. Prove that it is holomorphic if and only if its graph is a complex-analytic subset in $\mathbb{C}^2$.

Definition 2.17. Let (M, I_M) and (N, I_N) be almost complex manifolds, and $f: M \longrightarrow N$ a smooth map. It is called **holomorphic** if $f^*(\Lambda^{1,0}(N)) \subset \Lambda^{1,0}(M)$.

Exercise 2.18. Prove that a composition of holomorphic maps is holomorphic.

Hint. Identify $T^{1,0}(M)$ with the tangent bundle TM using the projection of $TM \subset TM \otimes_{\mathbb{R}} \mathbb{C}$ to $T^{1,0}M$ along $T^{0,1}M$. This defines a complex structure on $TM = (\Lambda^1(M))^*$. Prove that a map $f: M \longrightarrow N$ is holomorphic if and only if is differential is complex linear with respect to this complex structure on TM, TN.

Exercise 2.19. Let (M, I) be an almost complex manifold and $f: M \longrightarrow \mathbb{C}$ a function. Consider the standard almost complex structure on $\mathbb{C}$ Prove that f is a holomorphic function if and only if f is holomorphic as a map of almost complex manifolds.

Exercise 2.20. Let $M \subset \mathbb{C}^m, N \subset \mathbb{C}^n$ be open subsets, and $f: M \longrightarrow N$ a smooth map. Assume that for any holomorphic function on N, its pullback $f^*\phi$ is holomorphic on M. Prove that f is holomorphic.

Definition 2.18. An almost complex manifold (M, I) is called **integrable** if M ringed by the sheaf of holomorphic function is a complex manifold.

Exercise 2.21 (!). Let (M, I) and (N, I) be integrable almost complex manifolds. Prove that any holomorphic map $(M, I) \longrightarrow (N, I)$ defines a morphism of complex manifolds.

Hint. Use the previous exercise.

Exercise 2.22. Let $(M, \mathcal{O}_M)$ be a complex manifold. Prove that M admits a unique almost complex structure I such that $\mathcal{O}_M$ is the sheaf of holomorphic functions on (M, I).

Exercise 2.23 (*). Let (M, I) be an almost complex manifold such that for each $m \in M$ there exists a neighbourhood $U \ni m$ and a collection of holomorphic functions $f_1, ..., f_n$ on U such that their differentials $df_1, ..., df_n$ generate $\Lambda^{1,0}_m(M)$. Prove that the almost complex structure I is integrable.

Exercise 2.24 (**). Prove that a holomorphic function on an almost complex manifold cannot have a strict maximum.