Complex analytic spaces 3: Weierstrass preparation theorem

Rules: This is a class assignment for the next week. Exercises with [*] are extra hard and not necessary to follow the rest. Exercises with [!] are non-trivial, fundamental and necessary for further work.

3.1 Germs of holomorphic functions

Definition 3.1. Let $U, U' \subset \mathbb{C}^n$ be neighbourhoods of 0 and $f \in \mathcal{O}_U, f' \in \mathcal{O}_{U'}$ holomorphic functions. We say that f and f' have the same germ, $f \sim f'$ if $f\big|_{U\cap U'} = f'\big|_{U\cap U'}$. Clearly, $\sim$ gives an equivalence relation on the set of pairs $(U \ni 0, f \in \mathcal{O}_U)$. An equivalence class is called **germ of a holomorphic function**. We always consider germs as holomorphic functions defined in a neighbourhood of $0 \in \mathbb{C}^n$. The space of germs in 0 of holomorphic functions on $\mathbb{C}^n$ is denoted $\mathcal{O}_{0,\mathbb{C}^n}$ or $\mathcal{O}_n$. In the same way one defines the space of germs $\mathcal{O}_{x,M}$ of functions in $x \in M$, where M is a complex manifold.

Remark 3.1. Clearly, the equivalence relation $\sim$ is compatible with multiplication and addition. Therefore, $\mathcal{O}_{0,\mathbb{C}^n}$ is a ring.

Exercise 3.1. Let f be a holomorphic functions on a ball $B \subset \mathbb{C}^n$ which vanishes in an open subset $U \subset B$. Prove that $f = 0$.

Exercise 3.2. Let $U \subset V$ be connected open subsets of a complex manifold, and $H^0(\mathcal{O}_U)$, $H^0(\mathcal{O}_V)$ the rings of holomorphic functions on U, V. Prove that the restriction map $H^0(\mathcal{O}_U) \longrightarrow H^0(\mathcal{O}_V)$ is injective.

Definition 3.2. A ring $R' \supset R$ is called **finitely generated** over R if it is isomorphic to a quotient ring $R[t_1, ..., t_n]/I$ for some ideal $I \subset R[t_1, ..., t_n]$.

Exercise 3.3 (*). Prove that the ring $\mathcal{O}_n$ of germs of holomorphic functions is not finitely generated over $\mathbb{C}$ for any $n > 0$.

Definition 3.3. **Formal power series** in variables $t_1, ..., t_n$ is a sum

$$\sum_{i=0}^{\infty} P_i(t_1, ..., t_n),$$

where P_i are homogeneous polynomials of degree i. Addition of power series is defined componentwise, multiplication is defined via

$$\left(\sum_{i=0}^{\infty} P_i(t_1, ..., t_n)\right)\left(\sum_{i=0}^{\infty} Q_i(t_1, ..., t_n)\right) = \sum_{i=0}^{\infty} R_i(t_1, ..., t_n)$$

where $R_d(t_1, ..., t_n) = \sum_{i+j=d} P_i(t_1, ..., t_n) Q_j(t_1, ..., t_n)$.

Exercise 3.4. Prove that the space of power series is a ring.

Exercise 3.5. Construct an injective ring homomorphism from $\mathcal{O}_n$ to $\mathbb{C}[[t_1, ..., t_n]]$.

Exercise 3.6. Prove that $\mathcal{O}_n$ has no zero divisors.

Definition 3.4. A ring R is called **local** if it contains an ideal $I \subset R$ such that all elements $r \notin I$ are invertible.

Exercise 3.7. Prove that the ring $\mathcal{O}_n$ is local.

Exercise 3.8 (*). Prove that the ring $\mathbb{C}[[t_1, ..., t_n]]$ is not finitely generated over $\mathcal{O}_n \subset \mathbb{C}[[t_1, ..., t_n]]$.

3.2 Principal part of a germ of holomorphic function

Definition 3.5. Let $f \in \mathcal{O}_n$ be a germ of holomorphic function on $\mathbb{C}^n$. Write its Taylor series $f(z) = \sum_{i=0}^\infty P_i(t_1, ..., t_n)$, where P_i are homogeneous polynomials of degree i. We say that f has **zero of order (or of multiplicity) k in 0** if $P_0 = ... = P_{k-1} = 0$. In this situation **principal part** if the function f is the homogeneous polynomial P_k.

Exercise 3.9 (!). Let $\Phi(t_1, ..., t_n) = F_1(t_1, ..., t_n), ..., F_n(t_1, ..., t_n)$ be the holomorphic coordinate change around 0, with $F_i(0, ..., 0) = 0$, and $A := \left(\frac{dF_i}{dt_j}\right)$ its diferential. Prove that

a. For any germ $f \in \mathcal{O}_n$ which has 0 of multiplicity k, the function $\Phi^*(f)$ has zero of the same multiplicity.

b. The principal part of $\Phi^*(f)$ is obtained from the principal part of f by action of A.

Hint. Write Φ as a composition of A and a map

$$(t_1, ..., t_n) \longrightarrow G_1(t_1, ..., t_n), ..., G_n(t_1, ..., t_n),$$

where $G_i = t_i + P_i(t_1, ..., t_n)$, and all P_i have zeroes in 0 with multiplicity $\geqslant 2$.

Remark 3.2. For any germ $F \in \mathcal{O}_n$, the expression $F(0, z_n)$ denotes $F(0, 0, 0..., 0, z_n)$.

Exercise 3.10 (!). Let $F \in \mathcal{O}_n$ be a germ of holomorphic function with zero of multiplicity k. Prove that $\lim\limits_{z_n \to 0} \frac{F(0,z_n)}{z_n^k} = Q(0, ..., 1)$, where Q is the principal part of F.

Exercise 3.11 (!). Let Q be a non-zero homogeneous polynomial on $t_0, ..., t_n$, and $V(Q)$ its zero set, which we consider as a subset in $\mathbb{C}P^n$.

a. Prove that $\mathbb{C}P^n \backslash V(Q)$ is non-empty.

b. Prove that $V(Q) \subset \mathbb{C}P^n$ is a set of measure 0.

Exercise 3.12. Let $Q_1, ..., Q_n, ... \in \mathbb{C}[z_1, ..., z_{n+1}]$ be a countable set of homogeneous polynomials, and $Z_1, ..., Z_n, ... \subset \mathbb{C}P^n$ their zero sets. Prove that $\mathbb{C}P^n \backslash \bigcup Z_i$ is non-empty.

Exercise 3.13. Let $f_1, ..., f_n, ... \in \mathcal{O}_n$ be a countable collection of germs, which vanish with multiplicity $k_1, k_2, ...$ Prove that there exists a coordinate system $z_1, ..., z_n$, such that $\lim\limits_{z_n \to 0} \frac{f_i(0,z_n)}{z_n^{k_i}} \neq 0$ for all i.

Exercise 3.14 (**). Let $f \in \mathcal{O}_n$ be a germ with zero of multiplicity 2. Assume that its principal part is a non-degenerate quadratic form. Prove "the Morse lemma": for some coordinate system $z_1, ..., z_n$, the function f is written as $f = \sum z_i^2$.

Exercise 3.15 (**). Let $f \in \mathcal{O}_3$ be a germ of holomorphic function on $\mathbb{C}^3$. Prove that f is polynomial in appropriate coordinate system, or find a counterexample.

3.3 Newton formula

Definition 3.6. Let $e_i \in \mathbb{Z}[\alpha_1, ..., \alpha_n]$ be coefficients of a polynomial $t^n + e_1 t^{n-1} + ... + e_{n-1} t + e_n := \prod_{i=1}^n (t + \alpha_i)$. Then e_i are called **elementary symmetric polynomials** on α_i. **Newton polynomials** are $p_j := \sum_{i=1}^n \alpha_i^j$. **Complete homogeneous symmetric polynomial** of degree k is h_k obtained as a sum of all homogeneous monomials of degree k. The corressponding **generating functions** are formal series $E(t) := \sum_{i=0}^n e_i t^i$, $P(t) := \sum_{i=1}^\infty p_i t^i$, $H(t) := \sum_{i=0}^\infty h_i t^i \in \mathbb{Z}[\alpha_1, ..., \alpha_n][[t]]$.

Exercise 3.16. Prove that $H(t) = \prod_{i=1}^n \frac{1}{1-t\alpha_i}$.

Exercise 3.17. Prove that $E(t) = \prod_{i=1}^n (1+t\alpha_i)$.

Exercise 3.18. Prove that $H(t)E(-t) = 1$.

Exercise 3.19. Prove that $\frac{E'(-t)}{E(-t)} = -\sum_{i=1}^n \frac{\alpha_i}{1-t\alpha_i}$.

Exercise 3.20. Prove that $P(t) = -t\frac{E'(-t)}{E(-t)}$.

Exercise 3.21. Prove that p_i can be expressed as polynomials of e_i (with integer coefficients).

Exercise 3.22. Prove that h_i can be expressed as polynomials of e_i with integer coefficients. Prove that e_i can be expressed as polynomials of h_i with integer coefficients.

Exercise 3.23. (Newton formula) Prove that $ke_k = \sum_{i=1}^{k-1}(-1)^i e_{k-i}p_i$.

Hint. Use the formula $P(t) = -t\frac{E'(-t)}{E(-t)}$.

Exercise 3.24 (!). Prove that e_i are expressed as polynomials on p_i with rational coefficients.

Exercise 3.25 (*). Prove that $kh_k = \sum_{i=1}^k h_{k-i}p_i$.

3.4 Logarithmic derivative and Rouché theorem

Exercise 3.26 (!). Let f be a holomorphic function on a disk, non-zero everywhere on its boundary $\partial\Delta$, and $S_k(f) := \frac{1}{2\pi\sqrt{-1}}\int_{\partial\Delta}\frac{f'}{f}z^k dz$. Prove that $S_k(f) = \sum d_i\alpha_i^k$, where α_i are all zeros of f, and d_i their multiplicities.

Exercise 3.27. (Rouché theorem) Let f_t be a family of holomorphic functions on a disk Δ, continuously depending on a parameter $t \in \mathbb{R}$ and non-zero everywhere on its boundary $\partial\Delta$. Prove that the number of zeros of f_t in Δ is constant.

Hint. Use the previous exercise.

Exercise 3.28. Prove that all zeros of the polynomial $f(z) = z^5 + 3z^3 + 7$ belong to a disk $|z| \leqslant 2$.

Exercise 3.29. Prove that the equation $z + e^{-z} - 10 = 0$ has a unique solution with $\operatorname{Re} z > 0$.

Exercise 3.30 (!). Let $F(x,y) \subset \mathcal{O}_{\Delta\times\Delta}$ be a holomorphic function of two complex variables, having no zeros in the set $|x| = 1$, and $\phi(x)$ a holomorphic function on a unit disk $\Delta \subset \mathbb{C}$. Consider a function Φ mapping $y_0 \in \Delta$ to $\sum d_i\phi(\alpha_i)$, where α_i are all zeros of $F(x, y_0)$ in the disk $|x| \leqslant 1$, and d_i their multiplicities. Prove that Φ is holomorphic.

Exercise 3.31 (*). Let f_t be a continuous family of non-constant holomorphic functions on a disk, and $t \in [0,1]$ a real parameter. Let S be the set of all t such that f_t is injective. Prove that S is closed in $[0,1]$.

Hint. Use Rouché theorem.

3.5 Weierstrass preparation theorem

Definition 3.7. Let $z_1, ..., z_k$ be coordinates in $\mathbb{C}^k$. Denote the disk of radius r in $\mathbb{C}^k$ by $B_r(z_1, ...z_k)$.

Exercise 3.32. Let F be an analytic function in a neighbourhood of 0 in $\mathbb{C}^n$, such that $\lim\limits_{z_n \to 0} \frac{F(0,z_n)}{z_n^k} \neq 0, \infty$. Consider the projection map $\Pi: \ \mathbb{C}^n \longrightarrow \mathbb{C}^{n-1}$ $(z_1, ..., z_n) \longrightarrow (z_1, ..., z_{n-1})$.

a. (!) Prove that for an appropriate pair r, r', the restriction of F to the polydisk $\Delta(n-1,1) := B_r(z_1, ..., z_{n-1}) \times \Delta_{r'}(z_n)$ nowhere vanishes on the set $\Pi^{-1}(\partial \Delta_{r'}(z_n)$, where $\partial \Delta_{r'}(z_n)$ is the boundary of the disk.

b. (!) Prove that in this case the restriction of F to this polydisk has precisely k zeros $\alpha_1, ..., \alpha_k$ on each fiber of Π.

c. (!) Prove that $\sum_{i=1}^k \alpha_i^d$ is a holomorphic function on $B_r(z_1, ..., z_{n-1})$.

d. (!) Prove that any elementary symmetric polynomial on α_i gives a holomorphic function on $B_r(z_1, ..., z_{n-1})$.

Hint. For the last statement use the Newton formula to express the elementary symmetric polynomials through p_i.

Definition 3.8. A Weierstrass polynomial is a function $f \in \mathcal{O}_{n-1}[z_n]$, with the leading coefficient 1. Equivalently, a Weierstrass polynomial is $f(z_1, z_2, ..., z_n) = z_n^k + a_{k-1} z_n^{k-1} + ... + a_0$, with all $a_i \in \mathcal{O}_{n-1}$.

Exercise 3.33 (!). Let F be an analytic function in a neighbourhood of 0 in $\mathbb{C}^n$, such that $\lim\limits_{z_n \to 0} \frac{F(0,z_n)}{z_n^k} \neq 0, \infty$. Consider the projection map $\Pi: \ \mathbb{C}^n \longrightarrow \mathbb{C}^{n-1}$ $(z_1, ..., z_n) \longrightarrow (z_1, ..., z_{n-1})$, and let $P(z_n) \in \mathcal{O}_{n-1}[z_n]$ be a Weierstrass polynomial, which is expressed as $P(z_n) = \sum_{i=0}^k (-1)^{k-i} e_{k-i} z_n^i$, where e_i are elementary symmetric polynomial on the zeros $\alpha_1, ..., \alpha_k$ defined in the previous exercise. Prove that $F = P(z_n)u$, where u is a germ of an invertible holomorphic function.

Exercise 3.34 (!). Let $F \in \mathcal{O}_n$ be a germ of an analytic function.

a. Prove that in appropriate coordinate system, one has $F = uP(z_n)$, where $P(z_n)$ is a Weierstrass polynomial of degree k, such that $P(0, ..., 0, z_n) = z_n^k$.

b. Prove that for an appropriate coordinate system k is equal to the multiplicity of zero of F.

Definition 3.9. In this case, $P(z_n)$ is called **the Weierstrass polynomial** of F.

Exercise 3.35 (!). Let $F_1, ..., F_i, ... \in \mathcal{O}_n$ be a collection of germs of analytic functions. Prove that in appropriate coordinate system, all F_i can be written as $F_i = u_i P_i(z_n)$, where $P_i(z_n)$ is a Weierstrass polynomial.

Exercise 3.36. Consider a function $f(z, w) = wz^2 + (1 + w^2)z + w(1 + w^2)$ on $\mathbb{C}^2$. Compute its Weierstrass polynomial.

Hint. Express z through w by solving the quadratic equation $f(z, w) = 0$.