Complex variables 5: Ring of germs (Noetherianity, factoriality)

Rules: This is a class assignment for the next week. Exercises with [*] are extra hard and not necessary to follow the rest. Exercises with [!] are non-trivial, fundamental and necessary for further work.

Remark 5.1. All rings in the sequel are assumed commutative, associative and with unit.

5.1 Gauss lemma

Definition 5.1. An element a of a ring R is **invertible** if there exists $b \in R$ such that $ab = 1$. A non-invertible element $r \in R$ is called **prime** if for any divisor $r'|r$, either r' or r/r' is invertible.

Exercise 5.1. Prove that in the ring $\mathcal{O}_n$ of germs of holomorphic functions every element can be decomposed to a product of primes.

Exercise 5.2. Prove that in the ring $\mathcal{O}_M$ of holomorphic functions on an open subset $M \subset \mathbb{C}^n$, every element can be decomposed to a product of primes, or find a counterexample.

Definition 5.2. We say that a ring R is **factorial** if it has no zero divisors, any element of R has prime decomposition, and for any two decompositions $a = r_1 r_2 ... r_n = s_1 s_2 ... s_m$ to prime multipliers, these decompositions coincide up to the order and invertible multipliers.

Remark 5.2. Now we shall prove **Gauss lemma**: the polynomial ring $R[t]$ is factorial if R is factorial.

Exercise 5.3. Let R be a ring without zero divisors. Prove that the polynomial ring $R[t]$ has no zero divisors.

Definition 5.3. Let R be a factorial ring. A polynomial $P(t) \in R[t]$ is called **primitive** if the greatest common divisor (gcd) of its coefficients is 1.

Exercise 5.4 (!). Let $P_1(t), P_2(t) \in R[t]$ be primitive polynomials, and R factorial. Prove that the product $P_1(t)P_2(t)$ is also primitive.

Hint. Prove that $P_1(t)P_2(t)$ is non-zero modulo $p \in R$, if p is prime, and $P_1(t), P_2(t)$ are non-zero modulo p.

Exercise 5.5. Let R be a factorial ring, $P(t) \in R[t]$ primitive polynomial, and $rP(t) = r'P_1(t)P_2(t)$ decomposition of the polynomial $rP(t)$, where $r, r' \in R$ and $P_1(t), P_2(t) \in R[t]$ are primitive polynomials. Prove that r/r' is invertible.

Hint. Use the previous exercise.

Exercise 5.6. Let R be a factorial ring, and K its fraction field. Prove that every primitive polynomial $P(t) \in R[t]$, which is irreducible in $R[t]$, is also irreducible in $K[t]$.

Hint. Use the previous exercise.

Exercise 5.7. Prove **the Gauss lemma**: for any factorial ring R, the ring of polynomials $R[t]$ is also factorial.

Exercise 5.8. Let $f \in \mathcal{O}_{n-1}[z_n]$ be a Weierstrass polynomial of degree d, prime in the ring $\mathcal{O}_{n-1}[z_n]$, and satisfying $f(0,...,0,z_n) = z^d$. Prove that f is indecomposable in the ring $\mathcal{O}_n$.

Hint. Use the Weierstrass preparation theorem on the multipliers of f.

Exercise 5.9. Let $f = r_1 r_2 ... r_n = s_1 s_2 ... s_m$ be two prime decompositions in the ring $\mathcal{O}_n$. Prove that in some coordinate system, all s_i and r_i can be obtained as a product of an invertible function and Weierstrass polynomials of degree d satisfying $f(0,...,0,z_n) = z^d$.

Exercise 5.10. Prove that the ring $\mathcal{O}_1$ (germs of holomorphic functions in one variable) is factorial.

Exercise 5.11. Let $f \in \mathcal{O}_{n-1}[z_n]$ be a Weierstrass polynomial of degree d satisfying $f(0,...,0,z_n) = z^d$, and $f = r_1 r_2 ... r_n = s_1 s_2 ... s_m$ its prime decompositions. Assume that $\mathcal{O}_{n-1}$ is factorial. Prove that these two decompositions coincide up to the order and invertible multipliers

Hint. Use the Gauss lemma.

Exercise 5.12 (!). Prove that the ring $\mathcal{O}_n$ of germs is factorial.

Exercise 5.13 (*). Prove that the ring $\mathbb{C}[[t_1,...,t_n]]$ of formal power series is factorial.

5.2 Ascending chain condition

Definition 5.4. Let $(S, \prec)$ be a partially ordered set (poset). We say that S satisfies **ascending chain condition** if for any sequence $a_1 \preceq a_2 \preceq a_3 \preceq a_4 \preceq ...$ of elements of S, all a_i starting from some $N \gg 0$ coincide. The poset S satisfies **descending chain condition** if for any sequence $b_1 \succeq b_2 \succeq b_3 \succeq b_4 \succeq ...$ of elements of S, all b_i starting from some $N \gg 0$ coincide.

Definition 5.5. Let R be a ring, and S the set of all ideals in R, ordered by inclusion. We say that R is **Noetherian** if S satisfies the ascending chain condition, and **Artinian** if it satisfies the descending chain condition.

Exercise 5.14. Let R be a ring which has only one prime ideal. Prove that R is Artinian, or find a counterexample.

Exercise 5.15 (*). Let R be a ring which has only one prime ideal. Prove that R is Noetherian, or find a counterexample.

Remark 5.3. Consider the ring R as a module over itself. Clearly, submodules of R coincide with ideals of R.

Definition 5.6. An R-module M is **finitely generated over** R if there exists a finite collection $r_1, ..., r_n \in M$ such that $M = R \cdot r_1 + R \cdot r_2 + R \cdot r_3 + ...R \cdot r_n$. In this situation $r_1, ..., r_n$ are called **generators** of M. An ideal in R is called **finitely generated** if it is finitely generated as an R-module.

Exercise 5.16 (!). Prove that the ring R is Noetherian if and only if all its ideals are finitely generated.

Exercise 5.17. Prove that the rings $\mathbb{Z}$ and $\mathbb{C}[t]$ are Noetherian.

Exercise 5.18. Construct a ring which is not Artinian and not Noetherian.

Exercise 5.19 (*). Let M be a circle, and $C(M)$ the ring of continuous functions on M. Prove that $C(M)$ is non-Noetherian. Is it Artinian?

Exercise 5.20 (*). Let R be a Noetherian ring. Prove that R admits prime decomposition, or find a counterexample.

5.3 Noetherian modules

Definition 5.7. Let R be a ring. **Noetherian module** over R is an R-module which satisfies the ascending chain condition.

Exercise 5.21. Prove that any submodules and quotient modules of a Noeterian module are also Noetherian.

Exercise 5.22 (!). Prove that a ring R is Noetherian if and only if any ideal $I \subset R$ is finitely generated as an R-module.

Definition 5.8. Short exact sequence of R-modules is a sequence of R-modules and homomorphisms

$$0 \longrightarrow M_1 \xrightarrow{i} M_2 \xrightarrow{e} M_3 \longrightarrow 0$$

such i is injective, e surjective, $i \circ e = 0$, and $\ker e = \operatorname{im} i$.

Exercise 5.23 (!). Let $0 \longrightarrow M_1 \xrightarrow{i} M_2 \xrightarrow{e} M_3 \longrightarrow 0$ be an exact sequence of R-modules, where M_1 and M_3 are Noetherian. Prove that M_2 is also Noetherian.

Exercise 5.24 (*). Let $u : M \longrightarrow M$ be a surjective endomorphism of a Noetherian R-module. Prove that it is injective.

Hint. Use the ascending chain condition on a chain $\ker u \subset \ker u^2 \subset ...$.

Definition 5.9. An R-module M is called **cyclic** if it is isomorphic to R/I, where I is an ideal.

Exercise 5.25. Prove that an R-module is cyclic if and only if it is generated over R by one element $r \in M$.

Exercise 5.26. Let R be a Noetherian ring, and M a cyclic R-module. Prove that N is Noetherian.

Exercise 5.27 (!). Let M be an R-module. Prove that M is finitely generated if and only if it admits a filtration $0 = M_0 \subset M_1 \subset ... \subset M_n = M$ by R-submodules, and all subquotients M_i/M_{i-1} are cyclic.

Exercise 5.28. Let R be a Noetherian ring, and M an R-module. Prove that M is finitely generated if and only if it is Noetherian.

Hint. Use the induction by the number of generators and apply Exercise 5.23.

5.4 Lasker's theorem: the ring of germs is Noetherian

Exercise 5.29. Prove that the ring of holomorphic functions on a disk $\Delta \subset \mathbb{C}$ is non-Noetherian.

Exercise 5.30. Prove that the ring $\mathcal{O}_1$ of germs of holomorphic functions in one variable is Noetherian.

Exercise 5.31. Let $P(z, z_n) \in \mathcal{O}_{n-1}[z_n]$ – be a Weierstrass polynomial of degree k with $P(0, z_n) = z_n^k$, and a $(P) \subset \mathcal{O}_n$ the ideal generated by P. Prove that $\mathcal{O}_n/(P)$ is generated by $\mathcal{O}_{n-1}$ and $1, z_n, z_n^2, ..., z_n^{k-1}$.

Hint. Use the Weierstrass division theorem.

Exercise 5.32. Prove that the quotient $\mathcal{O}_n/(P)$ is finitely generated as an $\mathcal{O}_{n-1}$-module.

Exercise 5.33. Let $I \subset \mathcal{O}_n$ be an ideal in the ring of germs. Suppose that $\mathcal{O}_{n-1}$ is Noetherian. Let $P \in I$ be a Weierstrass polynomial of degree k with $P(0, z_n) = z_n^k$.

a. Prove that the image $I/(P)$ of I in $\mathcal{O}_n/(P)$ is finitely generated as an $\mathcal{O}_{n-1}$-module.

b. Let $\bar{r}_1, ..., \bar{r}_m$ be generators of $I/(P)$, considered as $\mathcal{O}_{n-1}$-module, and $r_1, ..., r_m$ their representatives over I. Prove that I is generated over $\mathcal{O}_n$ by P and $r_1, ...r_m$.

c. Prove that any ideal $I \subset \mathcal{O}_n$ is finitely generated as an $\mathcal{O}_n$-module.

Exercise 5.34 (!). Prove Lasker's theorem: the ring $\mathcal{O}_n$ is Noetherian.

Exercise 5.35 (*). Let A be the ring of rational functions on $\mathbb{C}^n$ which are holomorphic in 0. Consider A as a subring in $\mathcal{O}_n$, and let $R \subset \mathcal{O}_n$ be a subring containing A. Prove that R is Noetherian or find a counterexample.