Complex variables 7: Rückert Nullstellensatz

Rules: This is a class assignment for the next week. Exercises with [*] are extra hard and not necessary to follow the rest. Exercises with [!] are non-trivial, fundamental and necessary for further work.

7.1 Ideals in the ring of germs

Remark 7.1. In the sequel, we always assume that any ideal I in a ring R satisfies $I \neq R$; in other words, an ideal cannot contain unity.

Definition 7.1. The ring of germs of holomorphic functions on $\mathbb{C}^n$ in 0 is denoted $\mathcal{O}_n$. We consider these germs as functions (or power series) of $z_1, ..., z_n$. We consider $\mathcal{O}_d$ as a subring in $\mathcal{O}_n$, $n \geqslant d$. By convention, the functions in $\mathcal{O}_d$ depend on variables $z_1, ..., z_d$.

Definition 7.2. We say that $f \in \mathcal{O}_n$ has Weierstrass polynomial in the coordinates $z_1, ..., z_n$ if f has zero of order k in 0, and $\lim_{z_n \mapsto 0} \frac{f(0,...,0,z_n)}{z_n^k}$ is non-zero. In that case, the Weierstrass preparation theorem implies that $f = uP$, where u is invertible, and $P \in \mathcal{O}_{n-1}[z_n]$ is a monic polynomial of degree k.

Exercise 7.1. Let $J \subset \mathcal{O}_n$ be an ideal. Prove that for an appropriate reordering of coordinates, one would have $J_d = 0$ and $J_{d+i} \neq J_{d+i+1}$ for all $i \geqslant 0$.

Exercise 7.2. Let $J \subset \mathcal{O}_n$ be a prime ideal, and $J_k := J \cap \mathcal{O}_k \subset \mathcal{O}_n$. Denote by d the minimal number such that $J_{d+1} \neq 0$. Assume that $J_{d+i} \neq J_{d+i+1}$ for all $i \geqslant 0$.

a. (!) Prove that for an appropriate choice of coordinates, and all $k > d$, J_{k+1} is generated by J_k and Weierstrass polynomials in $\mathcal{O}_k[z_{k+1}]$.

b. Show that in this case $\mathcal{O}_n/J$ is a finitely generated $\mathcal{O}_k/J_k$-module for all $k > d$.

c. Show that the fraction field $k(\mathcal{O}_n/J)$ is a finite extension of $k(\mathcal{O}_d)$.

Definition 7.3. Such a coordinate system is called **a regular coordinate system for the ideal** J. Usually it iss written as $z_1, ..., z_d, z_{d+1}, ..., z_n$, where d is the largest number for which $J \cap \mathcal{O}_d = 0$. In this case, J is generated by $P_k(z_k)$, where $k = d+1, d+2, ..., n$.

Exercise 7.3 (*). Find an ideal $J \subset \mathcal{O}_2$ and a coordinate system z_1, z_2 such that $J_1 = 0$, but $\mathcal{O}_2/J$ is not finitely generated over $\mathcal{O}_1$.

7.2 Complex analytic subsets and their germs

Definition 7.4. A complex analytic subset (or "a complex analytic subvariety") of a complex manifold M is a closed subset $Z \subset M$ locally defined as a set of common zeros of a collection of holomorphic functions.

Definition 7.5. Let $Z_1, Z_2 \subset M$ be complex analytic subset. We say that Z_1 **is equal to** Z_2 **in a neighbourhood of** x, if $Z_1 \cap U = Z_2 \cap U$ for some neighbourhood $U \ni x$. Clearly, this defines an equivalence relation $Z_1 \sim Z_2$. **A germ of a complex analytic subset** in $x \in M$ is an equivalence class of complex analytic subsets $Z \subset U \ni x$ under this equivalence relation.

Remark 7.2. Let $J \subset \mathcal{O}_n$ be an ideal. The set Z_J of common zeros of J is a germ of complex subvarieties.

Exercise 7.4 (!). Let $J \subset \mathcal{O}_n$ be an ideal and Z_J its germ of zeros, considered as a germ of a variety. Assume that $J \cap \mathcal{O}_d = 0$. Let $\Pi_d : \mathbb{C}^n \longrightarrow \mathbb{C}^d$ be projection to the first d coordinates. Suppose that $z_1, ..., z_n$ is a regular coordinate system, $J_d = 0$, and $J_{d+1} \neq 0$. Prove that a preimage $\Pi_d^{-1}(x)$ of any point $x \in \mathbb{C}^d$ under the map $\Pi_d : Z_J \longrightarrow \mathbb{C}^d$ is finite.

Hint. Use induction in d.

Definition 7.6. A germ of a complex analytic subset Z in $x \in M$ is called **irreducible** if there is no non-trivial decomposition $Z = A_1 \cup A_2$ of Z to two germs of complex analytic subsets. **Irreducible component** of a germ Z is an irreducible complex analytic subset $Z_1 \subset Z$ such that the complement $Z \backslash Z_1$ is contained in a germ of complex analytic subsets which is strictly smaller than Z.

Exercise 7.5. Let Z be a germ of complex analytic subset, and $\mathcal{A}nn_Z \subset \mathcal{O}_n$ the ideal of functions vanishing in Z.

a. Prove that Z is irreducible $\Leftrightarrow$ $\mathcal{A}nn_Z$ is a prime ideal.

b. (!) Prove that Z is the union of its irreducible components.

Hint. Use Noetherianity of $\mathcal{O}_n$.

Exercise 7.6. Find an irreducible complex subvariety $S \subset \mathbb{C}^2$ such that its germ in 0 is not irreducible.

Exercise 7.7 (*). Let $S \subset \mathbb{C}^2$ be defined by an equation $x^n = y^m$, with m, n comprime. Prove that the germ of S in 0 is irreducible, or find a counterexample.

7.3 Finiteness theorem

Definition 7.7. Let $\phi : A \longrightarrow B$ be a ring homomorphism, such that B is finitely generated as an A-module. Then ϕ is called **a finite morphism**, and B **a finite** A**-algebra**.

Exercise 7.8. Let $\phi : A \longrightarrow B$ be a finite morphism, and $\mathfrak{p} \subset A$ a prime ideal.

a. (*) Prove that B contains a prime ideal $\mathfrak{p}'$ such that $\mathfrak{p} = \phi^{-1}(\mathfrak{p}')$.

b. (*) Prove that the set of such ideals is finite.

Hint. Let $A_\mathfrak{p}$ be a localization of A in $\mathfrak{p}$. Prove that $C := B \otimes_A (A_\mathfrak{p}/\mathfrak{p})$ is an Artinian algebra, and show that the ideals $\mathfrak{p}'$ such that $\mathfrak{p} = \phi^{-1}(\mathfrak{p}')$ are in 1-to-1 correspondence with the prime ideals in C.

Exercise 7.9. Let $\phi: A \longrightarrow B$ be a finite morphism of Noetherian rings, and $x \in B$. Prove that x is a root of a monic polynomial with coefficients in A.

Exercise 7.10. Let $P(x) \in A[x]$ be a monic polynomial. Prove that $A[x]/(P)$ is a finite A-algebra.

Exercise 7.11. Let $A_0 \subset A_1 \subset ... \subset A_n$ be a sequence of rings, such that each A_i is a finite algebra over A_{i-1}. Prove that A_n is a finite A_0-algebra.

Exercise 7.12 (!). (Finiteness theorem) Let $J \subset \mathcal{O}_n$ be an ideal, and $z_1, ..., z_n$ its regular coordinate system, such that $J_d = 0$, and $J_{d+1} \neq 0$. Prove that $\mathcal{O}_n/J$ is a finite $\mathcal{O}_d$-algebra.

Hint. Using the Weierstrass division theorem, prove that $\mathcal{O}_k/J_k$ is a finite $\mathcal{O}_{k-1}$-algebra, and then use Exercise 7.11.

Exercise 7.13. Let J be a prime ideal in $\mathcal{O}_n$, and $z_1, ..., z_n$ its regular coordinate system, such that $J_d = 0$, and $J_{d+1} \neq 0$. Prove that the fraction field $k(\mathcal{O}_n/J)$ is a finite extension of $k(\mathcal{O}_d)$.

Hint. Use the previous exercise.

Exercise 7.14. Let J be a prime ideal in $\mathcal{O}_n$, and $z_1, ..., z_n$ its regular coordinate system, such that $J_d = 0$, and $J_{d+1} \neq 0$. Denote by Z the germ of the set of common zeros of J.

a. (!) Prove that for any $u := \sum_{i=d+1}^n \lambda_i z_i$, the function $u \in \mathcal{O}_{d+1}$ satisfies a polynomial equation $P_u(u) = 0$, where $P_u[t] \subset \mathcal{O}_d[t]$ is a monic polynomial.

b. (!) Prove that for any linear function $u = \sum_{i=d+1}^n \lambda_i z_i$, the map $(z_1, ..., z_n) \xrightarrow{u} (z_1, ..., z_d, u)$ maps Z to a germ of a hypersurface $Z_u \subset \mathbb{C}^{d+1}$, defined by the equation $P_u(u) = 0$.

c. (!) Prove that for a general $u = \sum_{i=d+1}^n \lambda_i z_i$, the projection $Z \longrightarrow Z_u$ induces an isomorphism of the fraction fields $k(\mathcal{O}_{Z_u}) \longrightarrow k(\mathcal{O}_Z)$.

Hint. Use the finiteness theorem to show that $[k(\mathcal{O}_Z) : k(\mathcal{O}_{\mathcal{O}_d})]$ is a finite field extension, and apply the primitive element theorem to find a primitive $u = \sum_{i=d+1}^n \lambda_i z_i$.

7.4 Rückert Nullstellensatz

Exercise 7.15. Let Z_J be the zero set of an ideal $J \subset \mathcal{O}_n$ [1], and $z_1, ..., z_d, ..., z_n$ the regular coordinate system, $\mathcal{O}_d \cap J = 0$. Prove that a non-zero function $f \in \mathcal{O}_d$ does not vanish somewhere on Z_J.

Exercise 7.16. Let Z_J be the zero set of an ideal $J \subset \mathcal{O}_n$, $z_1, ..., z_d, ..., z_n$ the regular coordinate system, $\mathcal{O}_d \cap J = 0$, and $f \in \mathcal{O}_n/J$.

a. Prove that $P(f) = 0$ for some monic polynomial $P \in \mathcal{O}_d[t]$.

b. Suppose that J is prime, and that $f \in \mathcal{O}_n$ is non-zero somewhere on Z_J. Consider a polynomial of minimal degree $P = t^n + a_{n-1}t^{n-1} + ... + a_0 \in \mathcal{O}_d[t]$ such that $P(f) \in J$. Prove that $a_0 \neq 0$ somewhere on Z_J.

Hint. Use the previous exercise.

Exercise 7.17 (!). Let $J \subset \mathcal{O}_n$ be a prime ideal, and $f \in \mathcal{O}_n$ vanishes on the set $Z_J \subset \mathbb{C}^n$ of common zeros of J. Prove that $f \in J$.

Hint. Use the previous exercise.

Exercise 7.18 (!). Let $\Psi : J \mapsto Z_J$ associates to an ideal $J \subset \mathcal{O}_n$ the germ of the complex analytic set of its common zeros. Prove that Ψ defines a bijection between the set of prime ideals in $\mathcal{O}_n$ and the set of irreducible germs of complex analytic sets.

Hint. Use the previous exercise.

Definition 7.8. Let J be an ideal in a ring R. Define **radical** $\sqrt{J}$ as intersection of all prime ideals containing J. An ideal J is called **radical ideal** if $J = \sqrt{J}$.

Exercise 7.19. Prove that $a \in \sqrt{J}$ if and only if $a^n \in J$ for some $n > 0$.

Exercise 7.20. Let $J \subset \mathcal{O}_n$ be an ideal, and Z_J its zero set.

a. Prove that $Z_J = Z_{\sqrt{J}}$

b. Prove that $Z_J = Z_{\sqrt{J}} = \bigcup_{J' \in \mathfrak{P}} Z_{J'}$, where $\mathfrak{P}$ is the set of prime ideals in $\mathcal{O}_n$ containing J.

c. (!) Prove "Rückert Nullstellensatz": $f \in \mathcal{O}_n$ vanishes on Z_J if and only if $f \in \sqrt{J}$.

Exercise 7.21 (!). Let Z be a germ of a complex analytic set. Prove that Z is equal to the union of all its irreducible components, and there are finitely many of those components.

Hint. Use the noetherianlity of $\mathcal{O}_n$.

[1] The zero set of J is the set of its common zeros, considered as a germ of complex analytic sets.