Complex varieties 09: Divisors and the maximum principle

Rules: This is a class assignment for the next week. Exercises with [*] are extra hard and not necessary to follow the rest. Exercises with [!] are non-trivial, fundamental and necessary for further work.

9.1 Divisors

Definition 9.1. Let $Z \subset \mathbb{C}^n$ be a complex analytic subvariety. We call a point $z \in Z$ **smooth** if in a neighbourhood of z, the variety Z is a smooth submanifold of $\mathbb{C}^n$, and **singular** otherwise. A complex analytic subvariety $Z \subset \mathbb{C}^n$ is **smooth** if all its points are smooth. **Dimension** of Z in a smooth point $z \in Z$ is dimension of Z as a complex manifold. **Dimension of** Z is maximum of its dimension in all smooth points.

Definition 9.2. Let $Z, Z_1 \subset \mathbb{C}^n$ be germs of complex varieties in z, z_1, and $\phi : Z \longrightarrow Z_1$ map z to z_1. Recall that ϕ is called **a morphism of germs of complex varieties** if it is given by complex analytic functions in a neighbourhood of z. In this situation, the ring $\mathcal{O}_{Z,z}$ of germs of complex analytic functions on Z in z is a $\mathcal{O}_{Z_1,z_1}$-module, with $f \in \mathcal{O}_{Z_1,z_1}$ acting on $\mathcal{O}_{Z,z}$ by $f(a) = \phi^*(f)a$ for any $a \in \mathcal{O}_{Z,z}$. We say that Z **is finite over** Z_1 if $\mathcal{O}_{Z,z}$ is finitely generated as $\mathcal{O}_{Z_1,z_1}$-module, and **dominant** if the image of ϕ does not lie in a proper complex analytic subvariety of Z_1.

Exercise 9.1. Let Z be a complex variety of dimension d. Prove that any connected component of the set $Z \backslash Z_{\mathsf{sing}}$ of smooth points of Z has constant dimension.

Exercise 9.2. Let $Z \subset \mathbb{C}^n$ be an irreducible germ of a complex analytic variety defined by an ideal $J \subset \mathcal{O}_n$. Let d be the maximal number such that $\mathcal{O}_d \cap J = 0$ and $z_1, ..., z_d, ..., z_n$ a regular coordinate system. Prove that Z contains an open, dense subset which is d-dimensional.

Exercise 9.3. Let $\phi : Z \longrightarrow Z_1$ be a dominant morphism of germs of complex varieties of $\mathbb{C}^n$.

a. Let $z_1, ..., z_d, ..., z_n$ be regular coordinates on $Z_1 \subset \mathbb{C}^n$. Prove that a non-zero function $f \in \mathcal{O}_d$ does not vanish on Z.[1]

b. (!) Prove that there exists a regular coordinate system $z_1, ..., z_d, ..., z_n$ on Z_1, and a regular coordinate system $x_1, ..., x_{d+k}, ..., x_m$ on Z such that ϕ^* of a coordinate function is a coordinate function.

Exercise 9.4 (!). Let $\phi : Z \longrightarrow Z_1$ be a finite, dominant morphism. Prove that $\dim Z = \dim Z_1$.

Hint. Use the previous exercise.

Definition 9.3. Let X be a complex variety, $X_1, ..., X_n$ its irreducible components. **Divisor** (Cartier divisor) in a complex variety X is a subvariety $Z \subset X$, not containing any of X_i, and locally given by an equation $f = 0$ for some holomorphic function f.

[1] We interpret the ring $\mathcal{O}_{Z_1} \supset \mathcal{O}_d$ as a subring in $\mathcal{O}_z$ by identifying $f \in \mathcal{O}_{Z_1}$ and $\phi^* f \in \mathcal{O}_z$.

Exercise 9.5 (!). Let $Z \subset \mathbb{C}^n$ be a germ of a divisor. Prove that any irreducible component of Z is also a divisor.

Exercise 9.6. Let $Z \subset \mathbb{C}^n$ be a divisor. Prove that $\dim Z = n - 1$.

Exercise 9.7 (*). Find a germ of a singular variety X and a Cartier divisor $Y \subset X$ such that the irreducible components of Y are not Cartier divisors.

9.2 The variety of singular points of a variety

Exercise 9.8. Let $X \subset \mathbb{C}^n$ be a complex subvariety, and J its ideal. Prove that $x \in X$ is smooth of dimension k if and only if there exists functions $f_1, ..., f_k \in \mathcal{O}_{X,x}$ generating J in x such that the differentials $df_1, ..., df_k$ are linearly independent in x.

Hint. Use the inverse function theorem.

Exercise 9.9 (!). Prove that the set X_{sing} if singular points of X is complex analytic.

Hint. Use the previous exercise.

Exercise 9.10. Let $Y \subset X$ be a divisor which does not belong to the set X_{sing} of singular points of X. Prove that $\dim Y = \dim X - 1$.

Exercise 9.11 (!). Let X_{sing} be the set of singular points of a complex variety X. Prove that $\dim X_{\mathsf{sing}} < \dim X$.

Hint. Construct a finite, dominant morphism from X to $\mathbb{C}^d$, and prove that its differential is invertible outside of a divisor. Then apply Exercise 9.6 and Exercise 9.4.

Exercise 9.12. Let $Y \subset X$ be a divisor, and $\pi: X \longrightarrow \mathbb{C}^d$ a finite, dominant morphism. Prove that $\pi(Y)$ is contained in a divisor $D \subset \mathbb{C}^d$.

Hint. Use Exercise 9.25 below.

Exercise 9.13. Let $Y \subset X$ be a divisor. Prove that $\dim Y = \dim X - 1$.

Hint. Use the previous exercise and Exercise 9.6.

Exercise 9.14 (*). Let $Y \subset X$ be a divisor, and X_{sing} the set of singular points of X. Prove that if $Y \subset X_{\mathsf{sing}}$, then Y is a union of irreducible components of X_{sing}.

Exercise 9.15. Prove that any complex variety is not equal to a countable union of its divisors.

Hint. First check it for a smooth manifold.

Exercise 9.16 (!). Prove that a complement to a divisor is always open and dense.

Exercise 9.17. Let $X \subset Y$ be complex varieties. Prove that $\dim X \leqslant \dim Y$.

Exercise 9.18 (*). Let $X = \bigcup X_i$ be an irreducible decomposition of a complex variety, X_{sm} the set of smooth points of X, and f_i a collection of meromorphic functions on each X_i. Define f on X_{sm} writing $f = f_i$ on smooth points of each irreducible component. Prove that for each point $x \in X$, f can be expressed as a fraction of two holomorphic functions in an appropriate neighbourhood of x.

9.3 Maximum principle

Definition 9.4. An open map $\phi : X \longrightarrow Y$ is a continuous map of topological spaces such that $\phi(U)$ is open for any open subset $U \subset X$.

Exercise 9.19. Let f be a non-constant holomorphic function on a unit disk. Prove that the number of solutions of an equation $f(z) = c$ is constant in a sufficiently neighbourhood of 0.

Hint. Use Rouché theorem.

Exercise 9.20. Let f be a non-constant holomorphic function on a connected complex manifold of dimension 1. Prove that f is open.

Exercise 9.21. Let f be a non-constant holomorphic function on a smooth, connected complex manifold Z of dimension 1. Prove that f is an open map.

Exercise 9.22. Let f be a non-constant holomorphic function on a connected complex manifold Z (any dimension). Prove that f is open.

Hint. Use the previous exercise.

Exercise 9.23 (!). Let f be a holomorphic function on an irreducible complex variety Z of dimension 1. Suppose that $|f|$ has a local non-strict maximum at a point $z \in Z$. Prove that f is constant.

Hint. Use the previous exercise.

Exercise 9.24. Let $P(z_n) \in \mathcal{O}_{n-1}[z_n]$ be a Weierstrass polynomial, with zero of multiplicity k in 0, such that $P(0, 0, ..., 0, z_n) = z_n^k$, and $f \in \mathcal{O}_n$ a germ of holomorphic function.

a. Consider a neighbourhood of zero of form $\Delta_r(z_1, ..., z_{n-1}) \times \Delta_{r'}(z_n)$, obtained as a product of polydisks of radius r, r'. Prove that for appropriate $r, r' \ll 1$, the projection π to the first $n-1$ coordinates defines a ramified k-sheeted covering $Z \longrightarrow \mathbb{C}^{n-1}$, where Z is the set of zeros of $P(z_n)$.

b. (!) Let $z = z_1, ..., z_{n-1}$, and

$$\pi_* f(z) := \frac{1}{2\pi\sqrt{-1}} \int_{\partial \Delta_{r'}} \frac{P'(z_n)}{P(z_n)} f(z, z_n) dz_n.$$

Prove $\pi_* f(z) := \sum_{x \in \pi^{-1}(z)} f(x)$ (counted with multiplicities).

Exercise 9.25. Let $X \stackrel{\pi}{\longrightarrow} \mathbb{C}^d$ be a finite morphism of germs of complex varieties, f a holomorphic function on X, and $\pi_* f(z) := \sum_{x \in \pi^{-1}(z)} f(x)$.

a. (!) Prove that $\pi_* f$ is meromorphic.

b. Prove that any meromorphic, bounded function a polydisk is holomorphic.

c. (!) Prove that $\pi_* f$ is always holomorphic.

Hint. To prove (a), produce a meromorphic isomorphism between X and a complex analytic hypersurface $X' \subset \mathbb{C}^{d+1}$, and apply the previous exercise to the meromorphic function on X' which corresonds to f under the isomorphism $k(\mathcal{O}_X) \cong k(\mathcal{O}_{X'})$.

Definition 9.5. The function $\pi_* f$ is called **pushforward** of f under the finite map π.

Exercise 9.26. Let $Z \subset \mathbb{C}^n$ be a compact complex subvariety. Prove that Z is zero-dimensional.

Hint. Use the previous exercise.

Exercise 9.27. Let $\alpha_1, ..., \alpha_k$ be a collection of complex numbers, and α their average, $\alpha = \frac{1}{k}\sum_{i=1}^k \alpha_i$. Suppose that $|\alpha| \geqslant \max|\alpha_i|$. Prove that $\alpha = \alpha_i$, for all i.

Exercise 9.28. Let $X \xrightarrow{\pi} \mathbb{C}^d$ be a finite map of complex varieties, f a holomorphic function on X and $\pi_* f$ its pushforward. Assume that $|f|$ reaches its local maximum in $x \in X$.

a. Prove that $\pi_* f$ reaches its local maximum in $\pi(x)$.

b. Prove that $\pi_* f$ is constant.

Hint. Use the previous exercise.

Exercise 9.29. Construct a finite map $\pi : \mathbb{C} \longrightarrow \mathbb{C}$ and a non-constant holomorphic function f on $\mathbb{C}$ such that $\pi_* f = \mathsf{const}$.

Exercise 9.30. Suppose that $X \xrightarrow{\pi} \mathbb{C}^d$ is a finite map, f a holomorphic function on X, and $x \in X$ a point such that $\pi^{-1}(\pi(y)) = \{x\}$. Assume that $|f|$ has local maximum in x. Prove that f is constant.

Hint. Use Exercise 9.28 and Exercise 9.27.

Exercise 9.31. Prove **the maximum principle:** a non-constant holomorphic function on an irreducible complex variety cannot have a local maximum.

Hint. Use the previous exercise and Exercise 9.28.

Exercise 9.32. Construct a connected, non-irreducible complex variety X, and a non-constant holomorphic function on X which has a local maximum.

Exercise 9.33. Prove that a zero-dimensional complex subvariety $Z \subset \mathbb{C}^n$ is a discrete set.

Exercise 9.34 (*). Let $\{\alpha_i\}$ be a sequence of points in $\mathbb{C}^n$ converging to 0. Prove that there exists a complex curve containing all of them, or find a counterexample.