Complex varieties 11: Remmert rank

Rules: This is a class assignment for the next week. Exercises with [*] are extra hard and not necessary to follow the rest. Exercises with [!] are non-trivial, fundamental and necessary for further work.

11.1 Remmert rank

Exercise 11.1. (constant rank theorem)
Let $F: X \longrightarrow Y$ be a smooth map of smooth manifolds, where $\operatorname{rk} F := \dim \operatorname{im} dF$ is constant. Prove that each point $x \in X$ has a neighbourhood $U \ni x$ such that $F(U)$ is a smooth manifold of dimension $\operatorname{rk} F$, and the fibers $F^{-1}(z) \cap U$ are smooth submanifolds of dimension $\ker dF$.

Hint. Use the implicit function theorem.

Definition 11.1. A complex analytic space is **equidimensional** if it has constant dimension in each smooth point.

Exercise 11.2. Let $F: X \longrightarrow Y$ be a holomorphic map, proper and with finite fibers.

a. Prove that there exists a smooth point $x \in X$ such that $F(X)$ is smooth point, and $dF\Big|_{T_x X}$ is injective.

b. (!) Assume that F is surjective. Prove that $\dim X = \dim Y$.

Hint. Use the previous exercise.

Exercise 11.3 (!). Let $Z \subset \mathbb{C}^n$ be an equidimensional subvariety, intersecting a k-dimensional subspace $V \subset \mathbb{C}^n$ in a discrete set. Prove that $\dim Z \leqslant n-k$.

Hint. Prove that $\dim Z \leqslant \dim(Z \cap W) + 1$ for any hyperplane $W \subset \mathbb{C}^n$ and apply induction in dimension of Z.

Exercise 11.4. Let $F: \mathbb{C}^n \longrightarrow \mathbb{C}^m$ be a holomorphic submersion preserving 0, and $Z \subset \mathbb{C}^n$ a complex analytic subset of a neighbourhood of 0, such that $F^{-1}(0) \cap Z = 0$. Prove that

a. There exists a holomorphic coordinate system on $\mathbb{C}^n$, $\mathbb{C}^m$ in a neighbourhood of 0, such that F, written in these coordinates, is equal to a linear projection.

b. (!) $F\Big|_Z$ is proper on its image and has finite fibers in a sufficiently small neighbourhood of 0.

Hint. Use the previous exercise to show that F has finite fibers. Use the regular coordinates to prove that $F\Big|_Z$ is proper and has finite fibers in a small neighbourhood of 0.

Definition 11.2. Let $x \in X$ be a point in a complex analytic variety. **Dimension of X in x**, denoted $\dim(X,x)$ is the maximal dimension of irreducible components of X containing x.

Definition 11.3. Let $F: X \longrightarrow Y$ be a holomorphic morphism of complex varieties. Define **the Remmert rank of F in** $x \in X$ as $\operatorname{Rrk}_x F := \dim(X,x) - \dim(F^{-1}(F(x)),x)$.

Exercise 11.5 (!). Let $X \subset \mathbb{C}^n$ be a complex analytic subset, containing x, and $V \subset \mathbb{C}^n$ an affine subspace of codimension k, containing x. Let $F: X \longrightarrow Y$ be a holomorphic map which satisfies $\dim F^{-1}(F(x')) \cap V = 0$. Prove that $\operatorname{Rrk}_x F \geqslant \dim X - k$.

Hint. Use Exercise 11.3.

Exercise 11.6 (!). In assumptions of the previous exercise, prove that $F^{-1}(F(x')) \cap V$ is finite for any x' in a sufficiently small neighbourhood of x.

Hint. Use Exercise 11.4.

Exercise 11.7. Let $F: X \longrightarrow Y$ be a holomorphic morphism of complex analytic varieties such that $\operatorname{Rrk}_x F = 0$ for some $x \in X$. Prove that the Remmert rank of F vanishes in some neighbourhood of F.

Hint. Use Exercise 11.4.

Exercise 11.8. Let $F: X \longrightarrow Y$ be a holomorphic morphism of complex analytic spaces. Assume that $X \subset \mathbb{C}^n$ and $\operatorname{Rrk}_x F = k$. Prove that there exists an affine subspace $V \subset \mathbb{C}^n$ such that $x \in V$ and $\operatorname{Rrk}_x F\big|_{X \cap V} = \operatorname{Rrk}_x F$ and $\dim F^{-1}(F(x)) = 0$.

Exercise 11.9 (!). Prove that the function $x \mapsto \mathsf{rk}_x F$ is upper semicontinuous.

Hint. Use the previous exercise, Exercise 11.7 and Exercise 11.5.

Exercise 11.10 (!). Let $F: X \longrightarrow Y$ be a holomorphic morphism of equidimensional complex varieties. Prove that the Remmert rank $\mathsf{rk}_x F$ reaches its maximum on an open, dense subset of X.

Hint. Restrict F to the set of smooth points of X, and apply the constant rank theorem.

Exercise 11.11. Let $F: X \longrightarrow Y$ be a holomorphic morphism of complex varieties, and k the maximal rank of dF in all smooth points of X. Prove that k is the maximum of $\operatorname{Rrk}_x(F)$ over all points $x \in X$.

11.2 Structure of the set where the Remmert rank is maximal

Exercise 11.12. Let $F: X \longrightarrow Y$ be a holomorphic morpism of complex varieties, and $X' \subset X$ the set of all smooth points $x \in X$ such that the differential dF has maximal rank in x. Prove that X' is complex analytic.

Exercise 11.13. Let $F: X \longrightarrow Y$ be a holomorphic, bimeromorphic, proper morphism of complex varieties, and $Z \subset X$ a complex subvariety. Prove $F(Z)$ is complex analytic.

Hint. Show that $F(Z)$ can be given as a set of common zeros if a collection of meromorphic functions.

Exercise 11.14 (!). Let $F: X \longrightarrow Y$ be a dominant morphism of germs of d-dimensional complex varieties. Prove that there exist germs of varieties X_1, Y_1 bimeromorphic to X and Y, and a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & X_1 \\ {\scriptstyle F}\downarrow & & {\scriptstyle F_1}\downarrow \\ Y & \longrightarrow & Y_1 \end{array}$$

such that the horizontal lines are bimeromorphisms, and F_1 is finite.

Hint. Using the regular coordinates, construct bimeromorphic maps $X \longrightarrow X_u$ and $Y \longrightarrow Y_u$, where $\phi_X: X_u \longrightarrow C_d$, $\phi_Y: Y_u \longrightarrow \mathbb{C}_d$ are finite ramified coverings, and take for X_1 the image of the graph of F in $X_u \times Y_u$. Use the previous exercise to show that this image is complex analytic. Take for F_1 the restriction of the finite map $X_u \times Y_u \longrightarrow \mathbb{C}^d \times \mathbb{C}^d$ to X_1, and for Y_1 the diagonal in $\mathbb{C}^d \times \mathbb{C}^d$.

Exercise 11.15. In assumptions of the previous exercise:

a. Let $X_1 \subset X$ be the set of all $x \in X$ such that $F^{-1}(F(x))$ has positive dimension. Prove that X_1 is contained in the set X_e of all points $x \in X$ such that F is not invertible in a neighbourhood of x.

b. Prove that the image of X_e in Y_1 is complex analytic.

c. Deduce from this that $F(X_e)$ is complex analytic in Y.

d. Prove that $F^{-1}(F(X_e)) \neq X$.

Exercise 11.16 (!). Let $F: X \longrightarrow Y$ be a holomorphic morphism of complex varieties, and $X_1 \subset X$ the union of all fibers of F of positive dimension. Prove that X_1 is complex analytic in X.

Hint. Use the previous exercise to show that $X_1 \subset F^{-1}(F(X_e))$. Prove that $F^{-1}(F(X_e))$ is complex analytic and has smaller dimension than X. Complete the proof using the induction in $\dim X$.

Exercise 11.17 (!). Let $F: X \longrightarrow Y$ be a morphism of germs of complex varieties, and k is the maximum of Remmert rank $\mathsf{rk}_x(F)$ for all $x \in X$. Assume that X is a germ of a subvariety in $\mathbb{C}^n$, Prove that there exists a linear, surjective projection $G: \mathbb{C}^n \longrightarrow \mathbb{C}^k$ such that the Remmert rank of the map $F \times G: X \longrightarrow Y \times \mathbb{C}^k$ vanishes outside of a closed, nowehere dense subset of X.

Hint. Use the constant rank theorem.

Exercise 11.18 (!). Let $F: X \longrightarrow Y$ be a holomorphic morphism of complex varieties, and $Z \subset X$ the union of all fibers of non-minimal dimension. Prove that Z is complex analytic in X.

Hint. Use the previous exercise, and apply Exercise 11.16 to the map $F \times G: X \longrightarrow Y \times \mathbb{C}^k$.

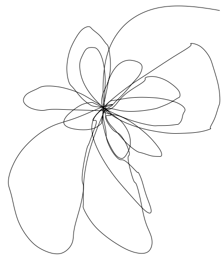
11.3 Remmert rank theorem

Exercise 11.19. Let $Z \subset \mathbb{C}^2$ be the zero set of non-constant holomorphic function. Prove that there exists a linear projection $\pi: \mathbb{C}^2 \longrightarrow \mathbb{C}$ such that in a sufficiently small neighbourhood of 0 the set $Z \cap \pi^{-1}(t)$ is finite for all $t \in \mathbb{C}$.

Hint. Use the Weierstrass preparation theorem.

Exercise 11.20 (!). Construct a holomorphic map $F: \mathbb{C} \longrightarrow \mathbb{C}^2$ such that its image is not contaned in any proper complex analytic subset.

Hint. Using polar coordinates, construct a real analytic function $f: \mathbb{R} \longrightarrow \mathbb{R}^2$ such that its image has infinite multiplicity in 0, as in the picture:



Extend this function to a holomorphic function $F:\ \mathbb{C} \longrightarrow \mathbb{C}^2$. Using the previous exercise, prove that a non-zero holomorphic functions cannot vanish on the image of $\mathbb{C}$.

Remark 11.1. From algebraic geometry it is known that the closure of the image of an algebraic manifold of dimension d under a regular map is at most d-dimensional. From the previous counterexample, it is clear that such an estimate for the dimension of the image of a holomorphic morphism $F:\ X \longrightarrow Y$ is wrong. Remmert rank theorem claims that locally in X such an estimate, nevertheless, exists: each of the segments of the line in the above picture is still a real analytic subvariety, and its complexification is complex analytic around the origin.

Exercise 11.21. Let $F:\ X \longrightarrow Y$ be a morphism of complex analytic varieties, and $X_0 \subset X$ the set of all smooth points in X where the rank of the differential dF is maximal.

a. Prove that X_0 is open and dense in X.

b. Prove that its complement is a closed complex analytic subset in X.

c. Prove that at any point $x \in X_0$, the rank of dF is equal to the Remmert rank $\operatorname{Rrk}_x F$.

d. Prove that the maximum of the Remmerk rank $\operatorname{Rrk}_x F$ is reached when $x \in X_0$

Exercise 11.22. Let $F:\ X \longrightarrow Y$ be a morphism of complex analytic varieties, and $X_0 \subset X$ the set of all smooth points in X where the rank of the differential dF is maximal. Denote by k the rank of dF in $x \in X_0$.

a. Prove that $F(X_0)$ is a countable union of smooth k-dimensional subvarieties of Y, not necessarily closed.

b. (!) **(Remmert rank theorem)**
Prove that $\operatorname{im} F$ is a countable union of complex subvarieties of Y of dimension $\leqslant k$ (not necessarily closed), and at least one of these subvarieties is k-dimensional.

Hint. Use induction in $\dim X$ and reduce the statement to Remmert rank theorem applied to the map $F\Big|_{X\setminus X_0}$.