Complex Varieties, 2023: final exam

Rules: Every student gets 10 exercises (randomly chosen from this problem set), the final grade is determined by the score. Exercises are worth 10 points, unless indicated otherwise. Please write down the solutions and bring them to exams. To pass the exam you are required to explain the solutions, using your notes. Please learn the proofs of all results you will be using on the way (you may put them in your notes). Please contact me in person or by email `verbit[]impa.br` when you are ready.

The final score N is obtained by summing up the points from the exam problems, adding 1 for each test exercise of the 4 sets of written exercises that you have been given. Marks: C when $30 \leqslant N < 50$, B when $50 \leqslant N < 60$, A when $60 \leqslant N < 80$, A+ when $N \geqslant 80$.

1 Holomorphic functions

Exercise 1.1 (10 pt). Let f be a function on a disk D such that $\int_{\partial S} f dz = 0$ for each square $S \subset D$ with sides parallel to the coordinate axes. Prove that f is holomorphic.

Exercise 1.2 (20 pt). Let $f: \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be a holomorphic function. Assume that the differential of f is non-degenerate at some point. Prove that f maps open sets to open sets, or find a counterexample.

Exercise 1.3 (20 pt). Let M be a connected complex variety admitting a non-constant holomophic function. Prove that the ring $\mathcal{O}_M$ of holomorphic functions on M is non-Noetherian.

Exercise 1.4 (30 pt). Let f be a continuous function on a complex manifold which is holomorphic in an open, dense subset. Prove that f is holomorphic or find a counterexample.

Exercise 1.5 (10 pt). Construct a bounded, holomorphic function in the domain

$$\{(x, y) \in \mathbb{C}^2 \ \ | \ \ |x| < 1, |y| < 1, \mathrm{Re}(x) > 0\}$$

which cannot be extended to any neighbourhood of $(0, 0)$.

2 The ring of germs of holomorphic functions

Definition 2.1. A local ring R with maximal ideal $\mathfrak{m}$ is called **strictly Henselian** if its residue field $R/\mathfrak{m}$ is algebraically closed, and every monic polynomial $P \in R[t]$ such that the image of P in $R/\mathfrak{m}[t]$ has no multiple roots, has a root in R.

Exercise 2.1 (10 pt). Prove that the ring of formal series $\mathbb{C}[[t_1, ..., t_n]]$ is strictly Henselian.

Exercise 2.2 (20 pt). Prove that the ring $\mathcal{O}_n$ of germs is strictly Henselian.

Exercise 2.3 (10 pt). Let X be germ of an n-dimensional irreducible complex variety, $D \subset X$ a germ of an irreducible subvariety of codimension 1, and $J \subset \mathcal{O}_X$ its ideal. Prove that J is principal or find a counterexample.

Exercise 2.4 (10 pt). Let G be a finite group which acts on $\mathbb{C}^n$ linearly and preserves a germ of a subvariety $Z \subset \mathbb{C}^n$ in 0. Prove that the ideal of Z in $\mathcal{O}_n$ is generated by G-invariant functions, or find a counterexample.

Exercise 2.5. Let $A: \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be a linear map with all eigenvalues which satisfy $|\alpha_i| > 1$, and $I \subset \mathcal{O}_n$ an A^*-invariant ideal. Prove that I is generated by polynomials

a. (20 pt) when A is diagonalizable

b. (30 pt) when A is any linear map.

3 Weierstrass polynomials and regular coordinates

Exercise 3.1 (10pt). Compute the Weierstrass polynomial of a function $f(w,z) = wz^2 + (1+w^2)z + w(1+w^2)$.

Exercise 3.2 (20 pt). Let $f \in \mathcal{O}_{n-1}[z_n]$ be a Weierstrass polynomial, satisfying $f = gh$, where $g, h \in \mathcal{O}_n$. Prove that there exists an invertible $u \in \mathcal{O}_n$ such that gu and hu^{-1} are Weierstrass polynomials in the same coordinate system.

Exercise 3.3 (10 pt). Let $V \subset \mathbb{C}^n$ be a germ of complex-analytic subset, defined by $k < n$ equations. Prove that there exists a germ of a holomorphic map $F: V \longrightarrow \mathbb{C}^k$ which is surjective to a germ of $\mathbb{C}^k$.

Exercise 3.4 (10 pt). Let f be a holomorphic function on a smooth complex manifold M, and $V(f)$ its zero set. Prove that each point $z \in V(f)$ has a neighbourhood U such that the intersection $V(f) \cap U$ is connected.

Exercise 3.5 (10 pt). Let $Z \subset \mathbb{C}^n$ be a submanifold, given as a zero set of an irreducible homogeneous polynomial. Prove that its germ in 0 is irreducible.

Exercise 3.6 (10 pt). Let Z be a germ of an irreducible complex-analytic subvariety of $\mathbb{C}^n$ of dimension $n-k$ in all smooth points. Prove that Z is an irreducible component of a germ of variety which is given by k equations.

4 Meromorphic maps

Exercise 4.1 (10 pt). Let X, Y be compact complex varieties, $U \subset X, V \subset Y$ open, dense subsets, and $F: U \longrightarrow V$ a holomorphic map. Prove that F can be extended to a meromorphic map if and only if the closure of the graph of F in $X \times Y$ is complex analytic.

Exercise 4.2 (10 pt). Let $f: X \longrightarrow Y$ be a holomorphic, bijective, continuous morphism of complex varieties. Prove that f is bimeromorphic.

Exercise 4.3 (10 pt). Let X be a complex variety, and f a meromorphic function on X which satisfies an equation $P(f) = 0$, where $P(t) \in \mathcal{O}_X[t]$ is a monic polynomial. Prove that f is locally bounded.

Exercise 4.4 (10 pt). Let M be a germ in 0 of an irreducible complex variety of dimension $\geqslant 2$, and f a holomorphic function on $M \backslash 0$. Prove that f can be extended to a meromorphic function on M.

Exercise 4.5 (10 pt). Consider a 2-dimensional torus $T := \mathbb{C}/\mathbb{Z}^2$ with the natural complex structure. Prove that T admits non-constant meromorphic functions.

5 Complex varieties

Exercise 5.1 (10 pt). Let G be a finite group. Construct a complex manifold M with $\pi_1(M) = G$.

Exercise 5.2 (10 pt). Prove that the set of smooth points in any irreducible complex variety is connected.

Exercise 5.3 (10 pt). Let f be a non-constant holomorphic function on a connected complex manifold, and $V(f)$ its zero set. Prove that the complement $U \backslash V_f$ is connected, but not simply connected.

Exercise 5.4 (20 pt). Find a simply connected, irreducible complex variety M with a point $m \in M$ such that the fundamental group of the complement $M \backslash \{m\}$ is infinite and non-Abelian.

Exercise 5.5 (10 pt). Consider a map $\Phi: \mathbb{C} \longrightarrow \mathbb{C}^2$, with $\Phi(z) = (z^2 - z, z^3 - z)$. Prove that the closure of the image of Φ is complex analytic in a neighbourhood of $(0, 0)$.

Exercise 5.6 (10 pt). Let X_1, X_2 be complex varieties. Consider the space $Z = X_1 \sim X_2$ obtained by the identification of a point $x_1 \in X_1$ with $x_2 \in X_2$. Prove that Z can be equipped with a structure of a complex variety in such a way that the natural maps $X_1 \hookrightarrow Z$ and $X_2 \hookrightarrow Z$ are holomorphic.