Complex analytic spaces

lecture 1: Cauchy theorem

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Complex structure on vector spaces

DEFINITION: Let V be a vector space over $\mathbb{R}$, and $I: V \longrightarrow V$ an automorphism which satisfies $I^2 = -\operatorname{Id}_V$. Such an automorphism is called **a complex structure operator** on V.

We extend the action of I on the tensor spaces $V \otimes V \otimes ... \otimes V \otimes V^* \otimes V^* \otimes ... \otimes V^*$ by multiplicativity: $I(v_1 \otimes ... \otimes w_1 \otimes ... \otimes w_n) = I(v_1) \otimes ... \otimes I(w_1) \otimes ... \otimes I(w_n)$.

Trivial observations:

1. **The eigenvalues α_i of I are $\pm\sqrt{-1}$.** Indeed, $\alpha_i^2 = -1$.

2. **V admits an I-invariant, positive definite scalar product ("metric") g.** Take any metric g_0, and let $g := g_0 + I(g_0)$.

3. **I is orthogonal for such g.**
Indeed, $g(Ix, Iy) = g_0(x, y) + g_0(Ix, Iy) = g(x, y)$.

4. **I diagonalizable over $\mathbb{C}$.** Indeed, any orthogonal matrix is diagonalizable.

5. **There are as many $\sqrt{-1}$-eigenvalues as there are $-\sqrt{-1}$-eigenvalues.**

Comples structure operator in coordinates

This implies that in an appropriate basis in $V \otimes_{\mathbb{R}} \mathbb{C}$, the almost complex structure operator is diagonal, as follows:

$$\left[\begin{array}{cccc|cccc}
\sqrt{-1} & & & & & & & \\
& \sqrt{-1} & & & & 0 & & \\
& & \ddots & & & & & \\
& & & \sqrt{-1} & & & & \\
\hline
& & & & -\sqrt{-1} & & & \\
& 0 & & & & -\sqrt{-1} & & \\
& & & & & & \ddots & \\
& & & & & & & -\sqrt{-1}
\end{array}\right]$$

We also obtain its normal form in a real basis:

$$\begin{bmatrix}
0 & -1 & & & & & & \\
1 & 0 & & & & & & \\
& & 0 & -1 & & & & \\
& & 1 & 0 & & & & \\
& & & & \ddots & & & \\
& & & & & \ddots & & \\
& & & & & & 0 & -1 \\
& & & & & & 1 & 0
\end{bmatrix}$$

The Grassmann algebra

DEFINITION: Let V be a vector space. Denote by $\Lambda^i V^*$ the space of antisymmetric polylinear i-forms on V^*, and let $\Lambda^* V := \bigoplus \Lambda^i V^*$. Denote by $T^{\otimes i} V^*$ the algebra of **all** polylinear i-forms on V^* ("tensor algebra"), and let $\mathsf{Alt}: T^{\otimes i} V^* \longrightarrow \Lambda^i V^*$ be **the antisymmetrization**,

$$\mathsf{Alt}(\eta)(x_1, ..., x_i) := \frac{1}{i!} \sum_{\sigma \in \Sigma_i} (-1)^{\tilde{\sigma}} \eta(x_{\sigma_1}, ..., x_{\sigma_i})$$

where Σ_i is the group of permutations, and $\tilde{\sigma} = 1$ for odd permutations, and 0 for even. Consider the multiplicative operation ("wedge-product") on $\Lambda^* V$, denoted by $\eta \wedge \nu := \mathsf{Alt}(\eta \otimes \nu)$. The space $\Lambda^* V^*$ with this operation is called **the Grassmann algebra of antisymmetric forms on V^***.

REMARK: It is an algebra of anti-commutative polynomials.

Properties of Grassmann algebra:

1. $\dim \Lambda^i V := \binom{\dim V}{i}$, $\dim \Lambda^* V = 2^{\dim V}$.

2. $\Lambda^*(V \oplus W) = \Lambda^*(V) \otimes \Lambda^*(W)$.

The Hodge decomposition in linear algebra

DEFINITION: Let (V, I) be a space equipped with a complex structure. **The Hodge decomposition** $V \otimes_{\mathbb{R}} \mathbb{C} := V^{1,0} \oplus V^{0,1}$ is defined in such a way that $V^{1,0}$ is a $\sqrt{-1}$-eigenspace of I, and $V^{0,1}$ a $-\sqrt{-1}$-eigenspace.

REMARK: Let $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$. The Grassmann algebra of skew-symmetric forms $\Lambda^n V_{\mathbb{C}} := \Lambda^n_{\mathbb{R}} V \otimes_{\mathbb{R}} C$ admits a decomposition

$$\Lambda^n V_{\mathbb{C}} = \bigoplus_{p+q=n} \Lambda^p V^{1,0} \otimes \Lambda^q V^{0,1}$$

We denote $\Lambda^p V^{1,0} \otimes \Lambda^q V^{0,1}$ by $\Lambda^{p,q} V$. The resulting decomposition $\Lambda^n V_{\mathbb{C}} = \bigoplus_{p+q=n} \Lambda^{p,q} V$ is called **the Hodge decomposition of the Grassmann algebra**.

De Rham algebra

DEFINITION: Let M be a smooth manifold. **A bundle of differential i-forms on M** is the bundle $\Lambda^i T^*M$ of antisymmetric i-forms on TM. It is denoted $\Lambda^i M$.

REMARK: $\Lambda^0 M = C^\infty M$.

DEFINITION: Let $\bigotimes_k T^*M \xrightarrow{\Pi} \Lambda^k M$ be the antisymmetrization map,

$$\Pi(\alpha)(x_1, ..., x_n) := \frac{1}{n!} \sum_{\sigma \in \mathrm{Sym}_n} (-1)^\sigma \alpha(x_{\sigma_1}, x_{\sigma_2}, ..., x_{\sigma_n}).$$

Define **the exterior multiplication** $\wedge : \Lambda^i M \times \Lambda^j M \longrightarrow \Lambda^{i+j} M$ as $\alpha \wedge \beta := \Pi(\alpha \otimes \beta)$, where $\alpha \otimes \beta$ is a section $\Lambda^i M \otimes \Lambda^j M \subset \bigotimes_{i+j} T^*M$ obtained as their tensor multiplication.

REMARK: The fiber of the bundle $\Lambda^* M$ at $x \in M$ **is identified with the Grassmann algebra $\Lambda^* T_x^* M$.** This identification is compatible with the Grassmann product.

Almost complex manifolds

DEFINITION: Let M be a smooth manifold. An **almost complex structure** is an operator $I: TM \longrightarrow TM$ which satisfies $I^2 = -\operatorname{Id}_{TM}$.

The eigenvalues of this operator are $\pm\sqrt{-1}$. The corresponding eigenvalue decomposition is denoted $TM = T^{0,1}M \oplus T^{1,0}(M)$.

DEFINITION: Let $\Lambda^{p,0}(M,I) := \Lambda^p_{C^\infty_{\mathbb{C}}(M)}(T^{1,0}M)^*$, $\Lambda^{0,p}(M,I) := \Lambda^p_{C^\infty_{\mathbb{C}}(M)}(T^{0,1}M)^*$, and $\Lambda^{p,q}(M,I) := \Lambda^{p,0}(M,I) \otimes_{C^\infty_{\mathbb{C}}(M)} \Lambda^{0,q}(M,I)$.

CLAIM:

$$\Lambda^n M \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p+q=n} \Lambda^{p,q}(M,I)$$

Proof: Same as for the vector spaces. ■

Complex manifolds

EXAMPLE: Let $M = \mathbb{C}^n$, with the complex coordinates $z_1, ..., z_n$ and real coordinates $x_i := \operatorname{Re}(z_i), y_i := \operatorname{Im}(z_i)$. **The standard almost complex structure** is defined as $I(dx_i) = dy_i$, $I(dy_i) = dx_i$,.

DEFINITION: A complex manifold is an almost complex manifold which is locally isomorphic to $\mathbb{C}^n$ with this complex structure.

REMARK: A 1-form $\alpha \in \Lambda^1(M, \mathbb{C})$ satisfies $\alpha(Ix) = \sqrt{-1}\,\alpha(x)$ if and only if $\alpha \in \Lambda^{1,0}(M)$. Therefore, **a function $f: M \longrightarrow \mathbb{C}$ is complex differentiable if and only if $df \in \Lambda^{1,0}(M)$.**

De Rham differential

THEOREM: There exists a unique operator $C^\infty M \xrightarrow{d} \Lambda^1 M \xrightarrow{d} \Lambda^2 M \xrightarrow{d} \Lambda^3 M \xrightarrow{d} ...$ **satisfying the following properties**

1. On functions, d is equal to the differential.
2. $d^2 = 0$
3. $d(\eta \wedge \xi) = d(\eta) \wedge \xi + (-1)^{\tilde\eta} \eta \wedge d(\xi)$, where $\tilde\eta = 0$ where $\eta \in \lambda^{2i} M$ is **an even form,** and $\eta \in \lambda^{2i+1} M$ is **odd.**

DEFINITION: The operator d is called **de Rham differential**.

DEFINITION: A form η is called **closed** if $d\eta = 0$, **exact** if $\eta \in \operatorname{im} d$. The group $\frac{\ker d}{\operatorname{im} d}$ is called **de Rham cohomology** of M.

THEOREM: (Stokes' theorem)
Let M be a compact manifold with boundary ∂M and $\alpha \in \Lambda^{\dim M - 1} M$ a differential form on M. **Then** $\int_M d\alpha = \int_{\partial M} \alpha$**.**

Holomorphic functions

DEFINITION: Let $U \subset \mathbb{C}^n$ be an open subset, and $f: U \longrightarrow \mathbb{C}$ a function of class C^1 (differentiable at least once). We say that f is **holomorphic** if the differential $df: T_xU \longrightarrow \mathbb{C}$ is complex linear at each $x \in U$.

REMARK: Clearly, f is holomorphic if and only if $df \in \Lambda^{1,0}(U)$, where $\Lambda^{1,0}(U)$ is the Hodge (1,0)-component of the de Rham algebra.

Taylor series decomposition for holomorphic functions in 1 variable is implied by the Cauchy formula: for any folomorphic function f in disc $\Delta \subset \mathbb{C}$,

$$\int_{\partial\Delta} \frac{f(z)dz}{z-a} = 2\pi\sqrt{-1}\, f(a),$$

where $a \in \Delta$ any point, and z coordinate on $\mathbb{C}$. Indeed, in this case,

$$2\pi\sqrt{-1}\, f(a) = \sum_{i \geqslant 0} a^i \int_{\partial\Delta} f(z)(z^{-1})^{i+1},$$

because $\frac{1}{z-a} = z^{-1} \sum_{i \geqslant 0} (az^{-1})^i$.

Cauchy formula in dimension 1

Let's prove Cauchy formula, using Stokes' theorem. Since the space $\Lambda^{1,0}\mathbb{C}$ is 1-dimensional, $df\wedge dz=0$ for any holomorphic function on $\mathbb{C}$. This gives

CLAIM: A function on a disc $\Delta\subset\mathbb{C}$ **is holomorphic if and only if the form $\eta:=f\,dz$ is closed** (that is, satisfies $d\eta=0$). ■

Now, let S_ε be a radius ε circle around a point $a\in\Delta$, Δ_ε its interior, and $\Delta_0:=\Delta\backslash\Delta_\varepsilon$. Stokes' theorem gives

$$0=\int_{\Delta_0}d\left(\frac{f(z)dz}{z-a}\right)=-\int_{S_\varepsilon}\frac{f(z)dz}{z-a}+\int_{\partial\Delta}\frac{f(z)dz}{z-a},$$

hence Cauchy formula would follow if we show that $\lim\limits_{\varepsilon\to 0}\int_{S_\varepsilon}\frac{f(z)dz}{z-a}=2\pi\sqrt{-1}f(a)$.

Assuming for simplicity $a=0$ and parametrizing the circle S_ε by $\varepsilon e^{\sqrt{-1}\,t}$, we obtain

$$\int_{S_\varepsilon}\frac{f(z)dz}{z}=\int_0^{2\pi}\frac{f(\varepsilon e^{\sqrt{-1}\,t})}{\varepsilon e^{\sqrt{-1}\,t}}d(\varepsilon e^{\sqrt{-1}\,t})=$$

$$=\int_0^{2\pi}\frac{f(\varepsilon e^{\sqrt{-1}\,t})}{\varepsilon e^{\sqrt{-1}\,t}}\sqrt{-1}\,\varepsilon e^{\sqrt{-1}\,t}dt=\int_0^{2\pi}f(\varepsilon e^{\sqrt{-1}\,t})\sqrt{-1}\,dt$$

as ε tends to 0, $f(\varepsilon e^{\sqrt{-1}\,t})$ tends to $f(0)$, and this integral goes to $2\pi\sqrt{-1}f(0)$.

Augustin-Louis Cauchy, 1789-1857

Baron Augustin-Louis Cauchy, 1789-1857

...In August 1833 Cauchy left Turin for Prague to become the science tutor of the thirteen-year-old Duke of Bordeaux, Henri d'Artois (1820-1883), the exiled Crown Prince and grandson of Charles X.[12] As a professor of the École Polytechnique, Cauchy had been a notoriously bad lecturer, assuming levels of understanding that only a few of his best students could reach, and cramming his allotted time with too much material. The young Duke had neither taste nor talent for either mathematics or science, so student and teacher were a perfect mismatch.

Augustin-Louis Cauchy, 1789-1857 (2)

...Although Cauchy took his mission very seriously, he did this with great clumsiness, and with surprising lack of authority over the Duke. During his civil engineering days, Cauchy once had been briefly in charge of repairing a few of the Parisian sewers, and he made the mistake of mentioning this to his pupil; with great malice, the young Duke went about saying Mister Cauchy started his career in the sewers of Paris. Cauchy's role as tutor lasted until the Duke became eighteen years old, in September 1838.[10] Cauchy did hardly any research during those five years, while the Duke acquired a lifelong dislike of mathematics. The only good that came out of this episode was Cauchy's promotion to baron, a title by which Cauchy set great store... (Wikipedia)

$\int_{\partial\Delta} f dz = 0$ for f holomorphic was proven in

A.-L. Cauchy. Mémoire sur les intégrales d'efinies, prises entre des limites imaginaires (Memoire on definite integrals taken between imaginary limits). De Bure, Paris, 1825.

$\int_{\partial\Delta} \frac{f dz}{z-a} = 2\pi\sqrt{-1}\, f(a)$ was proven in

A.-L. Cauchy. Sur la mecanique celeste et sur un nouveau calcul qui s'applique a un grand nombre de questions diverses etc [On Celestial Mechanics and on a new calculation which is applicable to a large number of diverse questions], presented to the Academy of Sciences of Turin, October 11. 1831.

Holomorphic functions on $\mathbb{C}^n$

THEOREM: Let $f: U \longrightarrow \mathbb{C}$ be a differentiable function on an open subset $U \subset \mathbb{C}^n$. **Then the following are equivalent.**
(1) f **is holomorphic**.
(2) For any complex affine line $L \in \mathbb{C}^n$, the restriction $f|_L = \mathbb{C}$ is **holomorphic as a function of one complex variable.**
(3) f is expressed as a sum of Taylor series around any point $(z_1, ..., z_n) \in U$: for all sufficiently small $t_1, ..., t_n$, one has $f(z_1 + t_1, z_2 + t_2, ..., z_n + t_n) = \sum_{i_1,...,i_n} a_{i_1,...,i_n} t_1^{i_1} t_2^{i_2} ... t_n^{i_n}$.

Proof: Equivalence of (1) and (2) is clear, because a restriction of $\theta \in \Lambda^{1,0}(M)$ to a line is a $(1,0)$-form on a line, and, conversely, if df is of type (1,0) on each complex line, it is of type (1,0) on TM, which is implied by the following linear-algebraic observation.

LEMMA: Let $\eta \in V^* \otimes \mathbb{C}$ be a complex-valued linear form on a real vector space (V, I) equipped with a complex structure I. **Then $\eta \in \Lambda^{1,0}(V)$ if and only if its restriction to any I-invariant 2-dimensional subspace L belongs to $\Lambda^{1,0}(L)$.**
EXERCISE: Prove it.

(3) clearly implies (1). (1) implies (3) by Cauchy formula (many variables), proven below.

Cauchy formula (many variables)

REMARK: Let $U \subset \mathbb{C}^n$ be an open subset, and $z_1, ..., z_n$ complex coordinates. Holomorphicity of $f: U \longrightarrow \mathbb{C}$ is equivalent to $df \in \Lambda^{1,0}(M)$, which is equivalent to $df \wedge dz_1 \wedge dz_1 \wedge ... \wedge dz_n = 0$. Denote the form $dz_1 \wedge dz_1 \wedge ... \wedge dz_n$ by Φ. We obtain that f **is holomorphic if and only if the form** $f\Phi$ **is closed**

THEOREM: (Cauchy formula in dimension n)
Let $\Delta \subset \mathbb{C}^n$ be a polydisc (product of discs) of radius 1, and $\alpha_1, ..., \alpha_n \in \Delta$ complex numbers. Denote by $S \subset \mathbb{C}^n$ the product of circles of radius 1 in variables $z_1, ..., z_n$:, $S = S_1(z_1) \times S_1(z_2) \times ... \times S_1(z_n)$. Let f be a holomorphic function in a polydisc. **Then** $\int_S V = (2\pi\sqrt{-1}\,)^n f(\alpha_1, ...\alpha_n)$**, where**

$$V = \frac{f\Phi}{(z_1 - \alpha_1)(z_2 - \alpha_2) \times ... \times (z_n - \alpha_n)}.$$

Proof. Step 1: Denote by Z the set $\bigcup_{i=1}^n \{(z_1, ..., z_n) \mid z_i = \alpha_i\}$. The complement of $\Delta \backslash Z$ is the set of definition of the closed differential form V. Let S_ε be the product of circles of radius ε with center in $\alpha_1, ..., \alpha_n$. Then $S, S_\varepsilon \subset \mathbb{C}^n \backslash Z$, and **the tori S, S_ε are homotopy equivalent in the domain $\mathbb{C}^n \backslash Z$, where V is closed. It remains to show that** $\lim_{\varepsilon \to 0} \int_{S_\varepsilon} V = (2\pi\sqrt{-1}\,)^n f(\alpha_1, ...\alpha_n)$**.**

Cauchy formula (many variables), part 2

THEOREM: (Cauchy formula in dimension n)
Let $\Delta \subset \mathbb{C}^n$ be a polydisc (product of discs) of radius 1, and $\alpha_1, ..., \alpha_n \in \Delta$ complex numbers. Denote by $S \subset \mathbb{C}^n$ the product of circles of radius 1 in variables $z_1, ..., z_n$:, $S = S_1(z_1) \times S_1(z_2) \times ... \times S_1(z_n)$. Let f be a holomorphic function in a polydisc. **Then $\int_S V = (2\pi\sqrt{-1}\,)^n f(\alpha_1, ...\alpha_n)$, where**

$$V = \frac{f\Phi}{(z_1 - \alpha_1)(z_2 - \alpha_2)...(z_n - \alpha_n)}.$$

Proof. Step 1: Let S_ε be a product of circles of radius ε with center in $\alpha_1, ..., \alpha_n$. **It remains to show that** $\lim_{\varepsilon \to 0} \int_{S_\varepsilon} V = (2\pi\sqrt{-1}\,)^n f(\alpha_1, ...\alpha_n)$.

Step 2: To simplify notation we set $\alpha_i = 0$. Parametrize S_ε by the cube $[0, 2\pi]^n$ using the map $t_1, ..., t_n \longrightarrow \varepsilon e^{\sqrt{-1}\,t_1}, ..., \varepsilon e^{\sqrt{-1}\,t_n}$. This gives

$$\int_{S_\varepsilon} V = \int_{S_\varepsilon} f(z) \frac{dz_1}{z_1} \wedge ... \wedge \frac{dz_n}{z_n} =$$

$$= \int_0^{2\pi} ... \int_0^{2\pi} \frac{f(\varepsilon e^{\sqrt{-1}\,t_1}, \varepsilon e^{\sqrt{-1}\,t_2}, ..., \varepsilon e^{\sqrt{-1}\,t_n})}{\varepsilon e^{\sqrt{-1}\,t_1} \varepsilon e^{\sqrt{-1}\,t_2} ... \varepsilon e^{\sqrt{-1}\,t_n}} \varepsilon^n d\left(e^{\sqrt{-1}\,t_1}\right) d\left(e^{\sqrt{-1}\,t_2}\right) ... d\left(e^{\sqrt{-1}\,t_n}\right) =$$

$$= (\sqrt{-1}\,)^n \int_0^{2\pi} ... \int_0^{2\pi} f(\varepsilon e^{\sqrt{-1}\,t_1}, ..., \varepsilon e^{\sqrt{-1}\,t_n}) dt_1 dt_2 ... dt_n,$$

which converges to $(2\pi\sqrt{-1}\,)^n f(0, ..., 0)$. ∎

Cauchy formula and Taylor expansion

REMARK: Cauchy formula implies that **holomorphic functions defined in a polydisc have Taylor expansion in this polydisc**. Indeed,

$$f(\alpha_1, ...\alpha_n) = \frac{1}{(2\pi\sqrt{-1}\,)^n}\int_S \frac{f dz_1 \wedge ... \wedge dz_n}{(z_1 - \alpha_1)(z_2 - \alpha_2) \times ... \times (z_n - \alpha_n)}$$

Take the Taylor expansion of $(z_i - \alpha_i)^{-1}$ using

$$\frac{1}{(z_i - \alpha_i)} = \frac{z_i^{-1}}{(1 - \alpha_i z_i^{-1})} = \sum_{l=0}^{\infty} \alpha_i^l z_i^{-l-1}.$$

Then

$$f(\alpha_1, ...\alpha_n) = \sum_{i_1=0}^{\infty} ... \sum_{i_n=0}^{\infty} \alpha_1^{i_1} .... \alpha_{i_n}^{i_n} \int_{S_\varepsilon} f(z_1, ..., z_n) z_1^{-i_1-1} ... z_n^{-i_n-1} dz_1 \wedge ... \wedge dz_n.$$