Complex analytic spaces

lecture 2: Sheaves

Misha Verbitsky

IMPA, sala 236

August 9, 2023

Sheaves of functions

DEFINITION: A **presheaf of functions** on a topological space M is a collection of subrings $\mathcal{F}(U) \subset C(U)$ in the ring $C(U)$ of all functions on U, for each open subset $U \subset M$, such that the restriction of every $\gamma \in \mathcal{F}(U)$ to an open subset $U_1 \subset U$ belongs to $\mathcal{F}(U_1)$.

DEFINITION: **An open cover** of a topological space X is a family of open sets $\{U_i\}$ such that $\bigcup_i U_i = X$.

DEFINITION: A presheaf of functions $\mathcal{F}$ is called **a sheaf of functions** if these subrings satisfy the following condition. Let $\{U_i\}$ be an open cover of an open subset $U \subset M$ (possibly infinite) and $f_i \in \mathcal{F}(U_i)$ a family of functions defined on the open sets of the cover and compatible on the pairwise intersections:

$$f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$$

for every pair of members of the cover. **Then there exists $f \in \mathcal{F}(U)$ such that f_i is the restriction of f to U_i for all i.**

Sheaves and presheaves of functions: examples

Examples of sheaves:

* Continuous functions

* Smooth functions, any differentiability class

* Real analytic functions

* Holomorphic functions

* Measurable functions

* Locally constant functions

Examples of presheaves which are not sheaves:

* Constant functions **(why?)**

* Bounded functions **(why?)**

* Integrable functions **(why?)**

Sheaves

REMARK: The definition of a sheaf below **is a more abstract version of the notion of "sheaf of functions"** defined above.

DEFINITION: A **presheaf** on a topological space M is a collection of vector spaces (or abelian groups) $\mathcal{F}(U)$, for each open subset $U \subset M$, together with **restriction maps** $R_{UW}\mathcal{F}(U) \longrightarrow \mathcal{F}(W)$ defined for each $W \subset U$, such that for any three open sets $W \subset V \subset U$, $R_{UW} = R_{UV} \circ R_{VW}$. Elements of $\mathcal{F}(U)$ are called **sections of $\mathcal{F}$ over U**, and the restriction map often denoted $f|_W$

DEFINITION: A presheaf $\mathcal{F}$ is called **a sheaf** if for any open set U and any cover $U = \bigcup U_I$ the following two conditions are satisfied.

1. Let $f \in \mathcal{F}(U)$ be a section of $\mathcal{F}$ on U such that its restriction to each U_i vanishes. **Then $f = 0$.**

2. Let $f_i \in \mathcal{F}(U_i)$ be a family of sections compatible on the pairwise intersections: $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for every pair of members of the cover. **Then there exists $f \in \mathcal{F}(U)$ such that f_i is the restriction of f to U_i for all i.**

Sheaves and manifolds

DEFINITION: A smooth manifold is a topological space equipped with a sheaf of functions which is locally isomorphic to an open ball in $\mathbb{R}^n$ with the sheaf of smooth functions on it.

DEFINITION: A complex manifold is a topological space equipped with a sheaf of functions which is locally isomorphic to an open ball in $\mathbb{C}^n$ with the sheaf of holomorphic functions on it.

REMARK: This is equivalent to the definition given in terms of charts and atlases. Indeed, the coordinate functions in one patch are holomorphic in another, hence the transition maps are holomorphic.

Today we are going to define a singular version of "complex manifold" – the **complex analytic space**.

Jean Leray

Jean Leray (7.11.1906 - 10.11.1998)
with grand-daughter Clothilde, 1990

Leray in Oflag XVIIA, by Haynes Miller

"The Second World War broke out in 1939 and J. Leray [then Professor at the Sorbonne and an officer in the French army] was made prisoner by the Germans in 1940. He spent the next five years in captivity in an officers' camp, Oflag XVIIA1 in Austria [not far from Salzburg]. With the help of some colleagues, he founded a university there, of which he became the Director ("recteur"). His major mathematical interests had been so far in analysis, on a variety of problems which, though theoretical, had their origins in, and potential applications to, technical problems in mechanics or fluid dynamics. Algebraic topology had been only a minor interest, geared to applications to analysis. Leray feared that if his competence as a "mechanic" ("mécanicien," his word) were known to the German authorities in the camp, he might be compelled to work for the German war machine, so he converted his minor interest to his major one, in fact to his essentially unique one, presented himself as a pure mathematician, and devoted himself mainly to algebraic topology." (A. Borel, "Jean Leray and algebraic topology")

Université en Captivité

Fig 2: Lieutenant Jean Leray, POW, became the rector of the University in Captivity. The picture on the right shows him with his Edelbach colleagues. Some would later join him at the Sorbonne or the Collège de France [Gaz2000].

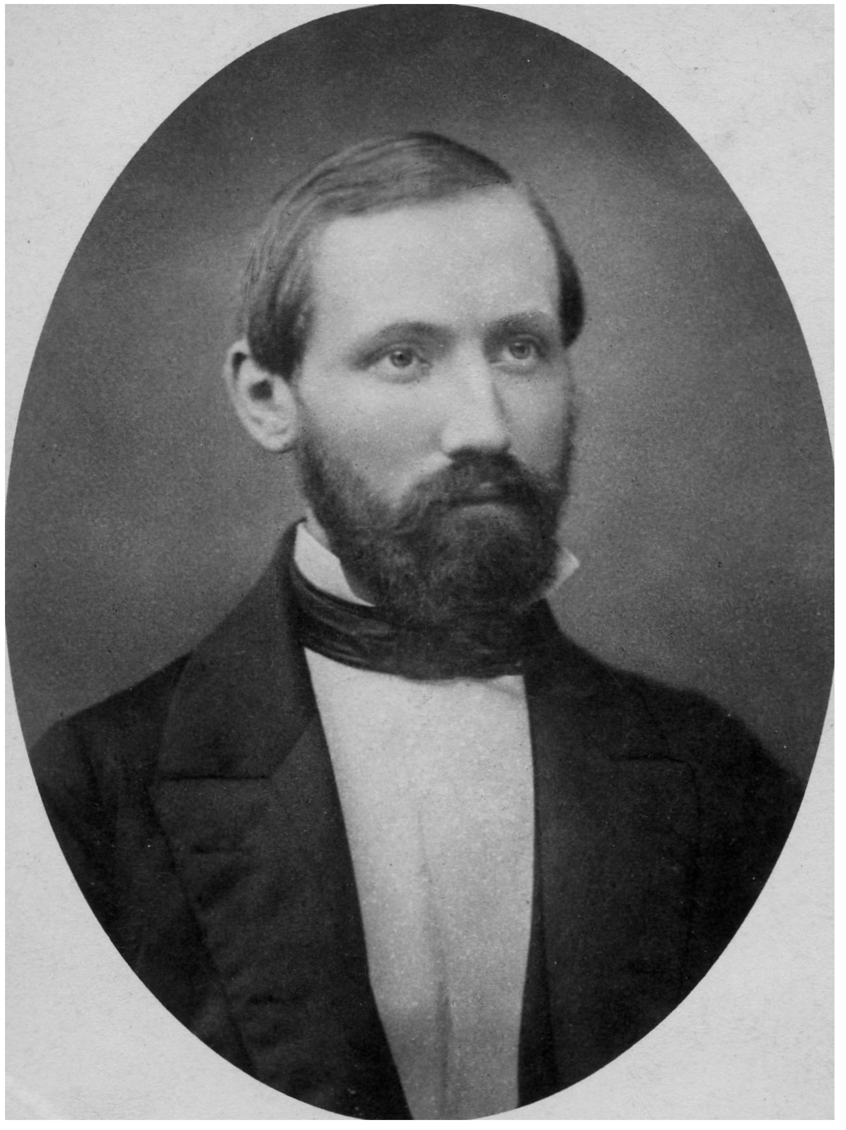
...From eight in the morning to eight in the evening, barrack 19 housed lectures on law and biology, on psychology and Arab language, on music and moral theology, on horse-raising (by a Polish fellow-officer, bien sur!), on public finances and on astronomy. The course on probability was given by lieutenant Jean Ville, who had published, just before the war, an ingenious elementary proof of von Neumann's minimax theorem. (Leray in Edelbach, by Anna Maria Sigmund, Peter Michor and Karl Sigmund)

Université en Captivité (2)

...The prisoners were encouraged to occupy their time productively. They formed a choir and a theatre group, and built their own sports ground, the Stade Pétain. One of the most popular activities were the lectures at the Université en Captivité, headed by Lieutenant Jean Leray, formerly a mathematics professor at the Université de Nancy. The University awarded almost 500 degrees, all of which were officially confirmed after the war. Leray lectured mainly on calculus and topology, concealing his expertise in fluid dynamics and mechanics since he feared being forced to work on German military projects. He also studied algebraic topology, publishing several papers after the war on spectral sequences and sheaf theory. Other notable figures of the University were the embryologist Étienne Wolff and the geologist François Ellenberger. The syllabus also included such subjects as law, biology, psychology, Arabic, music, moral theology, and astronomy (Wikipedia).

In Oflag XVIIA, Leray invented spectral sequences (given a formal definition by Serre in his thesis, 1950), sheaves and sheaf cohomology (Cartan, 1951, Grothendieck, 1955).

Bernhard Riemann (1826-1866)



Bernhard Riemann (1826-1866),
deutscher Mathematiker
date: c. 1850

Complex analytic spaces: the timeline

From the book "Coherent analytic sheaves" (H. Grauert, R. Remmert).

...From the very beginning it was clear that the notion of a complex manifold was not general enough. Already in $\mathbb{C}^3$ zero sets of quadratic polynomials like $w^2 - z_1 z_2$ are no complex manifolds but do have genuine singularities. In order to include such algebraic varieties one needs a cat- egory of "local models" larger than the category of open sets in $\mathbb{C}^n$. In 1951/52 two suggestions were made: H. BEHNKE and K. STEIN (cf. Math. Ann. 124, 1-16) chose finite analytically ramified coverings of domains of $\mathbb{C}^n$; H. CARTAN (cf. Sem. E.N.S. 1951/52, Exp.13, p.3) used special analytic sets in domains of $\mathbb{C}^n$. A characteristic feature of both definitions is that the resulting complex spaces X are irreducible everywhere and that holomorphic functions are continuous, i.e. the structure sheaf $\mathcal{O}_X$ of X is a subsheaf of the sheaf $\mathcal{C}_X$ of germs of continuous functions on X. Using the language and knowledge of today we may say that both definitions lead to the notion of a normal complex space. In 1954 CARTAN called these spaces "espaces analytiques généraux" (cf. Sém. E.N.S. 1953/54, Exp.6, p.8). But they were not general enough: soon it became clear that also spaces with reducible points had to be admitted. In 1955 SERRE in his GAGA paper (Ann. Inst. Fourier 6, 1-42) allowed all analytic sets in domains of $\mathbb{C}^n$ as local models. This seemed to be the end of the journey.

Complex analytic spaces: the timeline (2)

...However the refined study of fibers of holomorphic maps showed that SERRE'S category of complex spaces did not yet fit all purposes: the function $w = z^2$ *determines a 2-fold covering of* $\mathbb{C}$ *by itself with the origin 0 as double point: it is most natural to attach to 0 the 2-dimensional* $\mathbb{C}$*-algebra* $\mathcal{O}_0/\mathcal{O}_0 z^2$ *which has nilpotent elements* $\neq 0$*. Thus, in 1960, one finally was led to a notion of a complex space where holomorphic functions may be nilpotent and hence invisible for the geometric eye. Local models now are all analytic sets* A *in domains of* $\mathbb{C}^n$ together *with structure sheaves* $\mathcal{O}_A$ *which no longer need to be subsheaves of* $\mathcal{C}_X$ *(cf. [Gr2, p.9/10]). The way to this most general notion of a complex space had been paved before in Algebraic Geometry by A. GROTHENDIECK.*

[Gr2] Grauert, H.: Ein Theorem der analytischen Garbentheorie und die Modulriiume komplexer Strukturen, Publ. Inst. Hautes Etudes Sci. Vol. 5, 233-292 (1960)

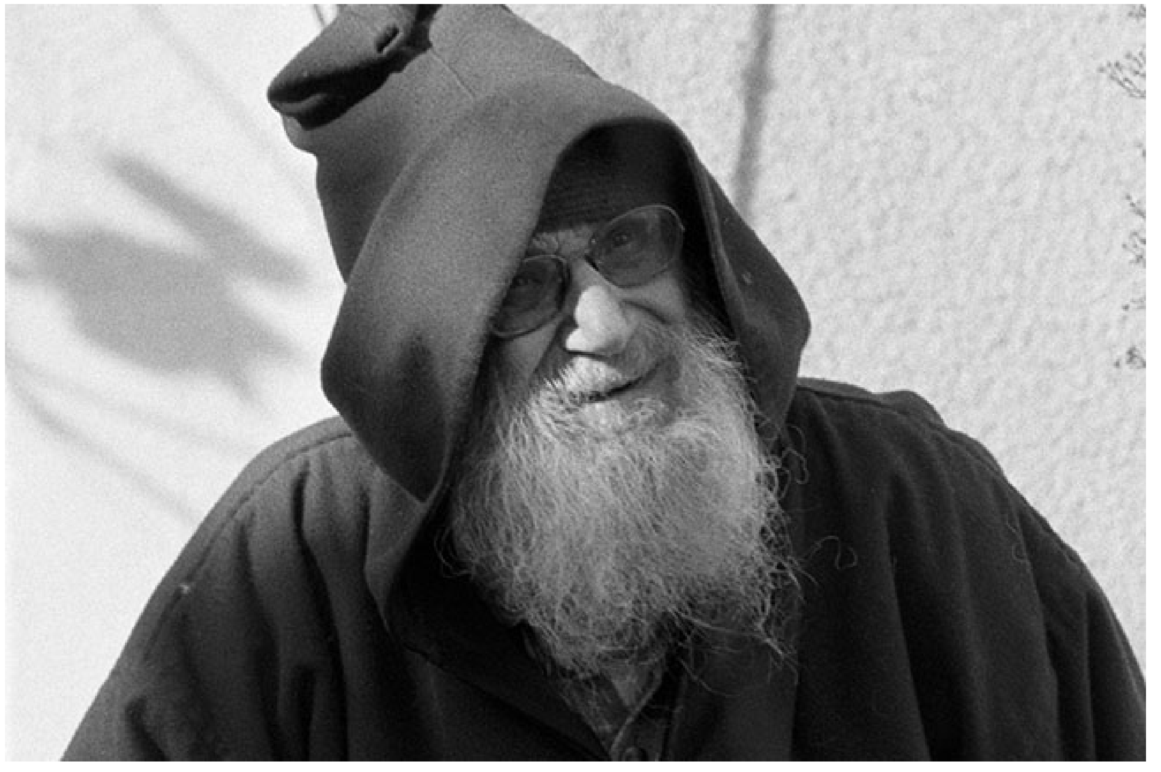
Hans Grauert

Hans Grauert in Bonn, 2000

(8.02.1930 - 4.09.2011)

Schemes

REMARK: I am bringing here the definition of the scheme in the name of the historical justice. In this course **it is never used,** except as a motivation.



Alexandre Grothendieck
(28 March 1928 - 13 November 2014)

Here and hereafter **all rings are assumed commutative and with unity.**

Spectrum of a ring

DEFINITION: Spectrum of a ring R is the set $\operatorname{Spec} R$ if its prime ideals.

DEFINITION: Let $J \subset R$ be an ideal, and $V(J) \subset \operatorname{Spec} R$ be the set of all prime ideals containing J. **Zariski topology** on $\operatorname{Spec} R$ is topology where all $V(J)$ (and only those) are closed. Clearly, $V(J_1) \cap V(J_2) = V(J_1 + J_2)$ and $V(J_1) \cup V(J_2) = V(J_1 J_2)$, hence finite unions and intersections of closed sets are closed.

Oscar Zariski
(1899 – 1986)

Affine schemes

DEFINITION: Let R be a ring, Spec(R) its spectum and $f \in R$. **Affine open set** is an open set $U_f := \operatorname{Spec} R \backslash V_{(f)}$. We identify u_f with Spec$(R[f^{-1}])$ (localization in f).

EXERCISE: Prove that **finite intersection of affine open sets is affine, $U_f \cap U_g = U_{fg}$.**

EXERCISE: Prove that **affine open sets give a base of Zariski topology.**

DEFINITION: The sheaf $\mathcal{O}$ of regular functions on $\operatorname{Spec} R$ is defined by $\mathcal{O}\big|_{U_f} = R[f^{-1}]$.

EXERCISE: Prove that $\mathcal{O}\big|_{U_f} = R[f^{-1}]$ **is sufficient to define a sheaf,** which is reconstructed uniquely from this property.

DEFINITION: A scheme is a ringed space $(M, \mathcal{O})$, which is locally isomorphic to an affine scheme with the sheaf of regular functions. In this situation sheaf $\mathcal{O}$ is called **the structure sheaf** of the scheme, or **the sheaf of regular functions**.

REMARK: The structure sheaf **may contain nilpotents.**

Complex analytic spaces

DEFINITION: Let B be an open ball in $\mathbb{C}^n$, and $H^0(\mathcal{O}_B)$ the ring of holomorphic functions, and $I \subset H^0(\mathcal{O}_B)$ an ideal. We define a presheaf $\mathcal{O}_B/I$ on B as follows: for any open subset $U \subset B$ define $H^0((\mathcal{O}_B/I)|_U) := (H^0(\mathcal{O}_B)/I) \otimes_{\mathcal{O}_B} \mathcal{O}_U$. It is not hard to see that this is a sheaf **(prove it)**.

DEFINITION: **A complex analytic space** is a ringed space which is locally isomorphic to $(V(I), \mathcal{O}_B/I)$ where $V(I)$ is the set of all $x \in B$ such that $f(x) = 0$ for all $f \in I$.

REMARK: Just as for schemes, **the structure sheaf of a complex analytic space might contain nilpotents.**

EXERCISE: Let $\Delta \subset \mathbb{C}$ be a disc and $I = (z^2) \subset \mathcal{O}_\Delta$. **Describe the space** $(\operatorname{Ann}(I), \mathcal{O}_\Delta/I)$**.**

DEFINITION: **A complex variety** is a complex analytic space without nilpotents in the structure sheaf.

In this course we will work only with complex varieties.