Complex analytic spaces

lecture 3: Germs of functions and Weierstrass preparation theorem

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Complex analytic subsets in $\mathbb{C}^n$

DEFINITION: Let $U \subset \mathbb{C}^n$ be an open set. **A complex analytic subset** $Z \subset U$ is a set of common zeroes of a finite family of holomorphic functions. **A holomorphic function** on Z is a restriction of a holomorphic function defined in a neighbourhood of Z. A **complex analytic subvariety** of U is Z equipped with the sheaf of holomorphic functions.

DEFINITION: Two complex analytic subvarieties Z_1, Z_2 are **isomorphic** if they are isomorphic as ringed spaces, that is, there exists a homeomorphism $Z_1 \longrightarrow Z_2$ taking any holomorphic function (locally defined or global) on Z_1 to a holomorphic function on Z_2 and a holomorphic function on Z_2 to a holomorphic function on Z_1.

Morphism of complex varieties

DEFINITION: A morphism of complex varieties is a continuous map $Z_1 \longrightarrow Z_2$ such that a pullback of a locally defined holomorphic function on Z_2 is holomorphic on Z_1.

PROPOSITION: Consider complex subvarieties $Z_1 \subset U_1 \subset \mathbb{C}^n$, $Z_2 \subset U_2 \subset \mathbb{C}^m$ and a morphism $\varphi: (Z_1, \mathcal{O}_{Z_1}) \longrightarrow (Z_2, \mathcal{O}_{Z_2})$. Then **there exist an open neighbourhood $W_1 \supset Z_1$ such that φ can be restricted to a holomorphic map $W_1 \longrightarrow U_2$.**

Proof: The pullback map φ^* takes the coordinate functions $z_1, ..., z_n$ on $U_2 \subset \mathbb{C}^n$ restricted to Z_2 to holomorphic functions $\alpha_1, ..., \alpha_n$ on Z_1. These functions are obtained as restrictions from a certain neighbourhood of Z_1, denoted as W_1. Clearly, for any $z \in Z_1$, the number $\alpha_i(z)$ is the i-th coordinate of $\varphi(z) \in \mathbb{C}^n$. Therefore, the map $(\alpha_1, ..., \alpha_n): W_1 \longrightarrow U_2$ taking $z \in W_1$ to $(\alpha_1(z), ..., \alpha_n(z))$ is equal to φ on Z_1. ■

The ring of germs of holomorphic function

Let $x \in X$ be a point in a complex analytic variety, and $U_1, U_2 \ni x$ two connected open neighbourhoods and $f_1 \in H^0(\mathcal{O}_{U_1})$, $f_2 \in H^0(\mathcal{O}_{U_2})$. We write $f_1 \sim f_2$ if these functions are equal on $U_1 \cap U_2$. Clearly, this is an equivalence relation which is compatible with the ring structure.

DEFINITION: The set of equivalence classes of functions $f \in H^0(\mathcal{O}_U)$, for all connected open sets $U \ni x$, is called **the ring of germs of holomorphic functions in x**, denoted by $\mathcal{O}_{X,x}$ or $\mathcal{O}_x$.

EXERCISE: ("the principle of analytic continuation")
Let f be a holomorphic function on a connected open subset $U \subset \mathbb{C}^n$. Suppose that $f = 0$ on an open subset of U. **Prove that $f = 0$ on U.**

COROLLARY: Let X be a smooth, connected complex manifold. **Then the natural restriction map $H^0(\mathcal{O}_X) \longrightarrow \mathcal{O}_{X,x}$ is injective**.

EXERCISE: Prove this.

Formal power series

DEFINITION: A formal power series of variables $t_1, ..., t_n$ is a sum $\sum_{i=0}^\infty P_i(t_1, ..., t_n)$, where all P_i are homogeneous polynomials of degree i. Addition of formal series is defined componentwise, the product is defined as on polynomials, using

$$\left(\sum_{i=0}^\infty P_i(t_1, ..., t_n)\right)\left(\sum_{i=0}^\infty Q_i(t_1, ..., t_n)\right) = \sum_{i=0}^\infty R_i(t_1, ..., t_n)$$

where $R_d(t_1, ..., t_n) = \sum_{i+j=d} P_i(t_1, ..., t_n) Q_j(t_1, ..., t_n)$.

EXERCISE: Prove that these operations **define a structure of a ring on the space of formal power series**. The ring of formal power series is denoted $\mathbb{C}[[t_1, ..., t_n]]$.

DEFINITION: A ring R is called **local** if it contains an ideal $I \subset R$ such that any element $a \in R \backslash I$ is invertible in I.

EXERCISE: Prove that **the ring of formal power series is local.**

CLAIM: The ring $\mathcal{O}_{X,x}$ of germs of holomorphic functions is local.
Proof: Let $I \subset \mathcal{O}_{X,x}$ be the ideal of all functions vanishing in x. Given $f \in H^0(\mathcal{O}_U)$ such that $f(x) \neq 0$, consider $W := \{y \in U \ \ | \ \ f(y) \neq 0\}$. By definition, W is open, and f is invertible in W. ■

Taylor series and germs of holomorphic functions

DEFINITION: Let M be a complex manifold, $x \in M$ a point, and $t_1, ..., t_n$ complex coordinates in its neighbourhood, with $t_i(x) = 0$. To any germ $f \in \mathcal{O}_{M,x}$, we attach a formal power series

$$\sum_{r=0}^{\infty} \sum_{\substack{i_1+i_2+... \\ ...+i_n=r}} \frac{d^r f}{dt_1^{i_1} dt_2^{i_2} ... dt_n^{i_n}} \cdot \frac{t_1^{i_1} t_2^{i_2} ... t_n^{i_n}}{i_1! i_2! ... i_n!}$$

This formal power series is called **the Taylor power series of f**.

PROPOSITION: The Taylor power series of a holomorphic function f **converges to f in a neighbourhoof of x.**

Proof: Lecture 1. ■

CLAIM: The operation of taking the Taylor power series **defines a ring homomorphism $\mathcal{O}_{M,x} \longrightarrow \mathbb{C}[[t_1, ..., t_n]]$.**

EXERCISE: Prove this.

Order of zero of a holomorphic function

DEFINITION: Let f be a holomorphic function on $U \subset \mathbb{C}^n$, vanishing in $0 \in U$, and $f(z) = \sum_{i=0}^\infty P_i(t_1, ..., t_n)$ be its Taylor series, where P_i are homogeneous polynomials of degree i. We say that f **has zero of order k in 0**, or **zero of multiplicity k**, if $P_0 = ... = P_{k-1} = 0$.

PROPOSITION: The order of zero of f in 0 is invariant under holomorphic changes of coordinates preserving 0.

Proof: Let $(t_1, ..., t_n) \stackrel{F}{\mapsto} F_1(t_1, ..., t_n), ..., F_n(t_1, ..., t_n)$ be a holomorphic diffeomorphism defined in a neighbourhood of 0 and mapping 0 to 0. Here F_i are holomorphic functions such that the matrix $dF = \left(\frac{dF_i}{dt_j}\right)$ is non-degenerate. Composing F with a linear coordinate change, we can assume that $F_i(t_1, ..., t_n) = t_i + G_i$, where G_i is a sum of homogeneous polynomials of degree $\geqslant 2$. Clearly, **the composition of a homogeneous polynomial Q of degree d and F is $Q + A$, where A is a sum of homogeneous polynomials of degree $> d$.** Therefore, the order of zero of $F^*(f)$ is equal to the order of zero of f. ■

Principal part of a holomorphic function

DEFINITION: Let $f(t_1, ..., t_n)$ be a holomorphic function which has zero of order d in 0. The **principal part** f is a homogeneous polynomial $Q(t_1, ..., t_n)$ of degree d such that $f - Q$ has zero of order $\geqslant d + 1$.

REMARK: From the previous proof it also follows that **the principal part of f is invariant under the coordinate change**. More precisely, a holomorphic coordinate change acts on the principal part of f as a linear map $t_i \longrightarrow \sum_j a_{ij} t_j$ where $a_{ij} \in \mathbb{C}$.

EXERCISE 1: Let f be a holomorphic function on $U \subset \mathbb{C}^n$ which has in 0 a zero of order k. Prove that **for any coordinate system centered in 0, the limit $\lim\limits_{z_n \to 0} \frac{f(0,...,0,z_n)}{z_n^k}$ is finite.**

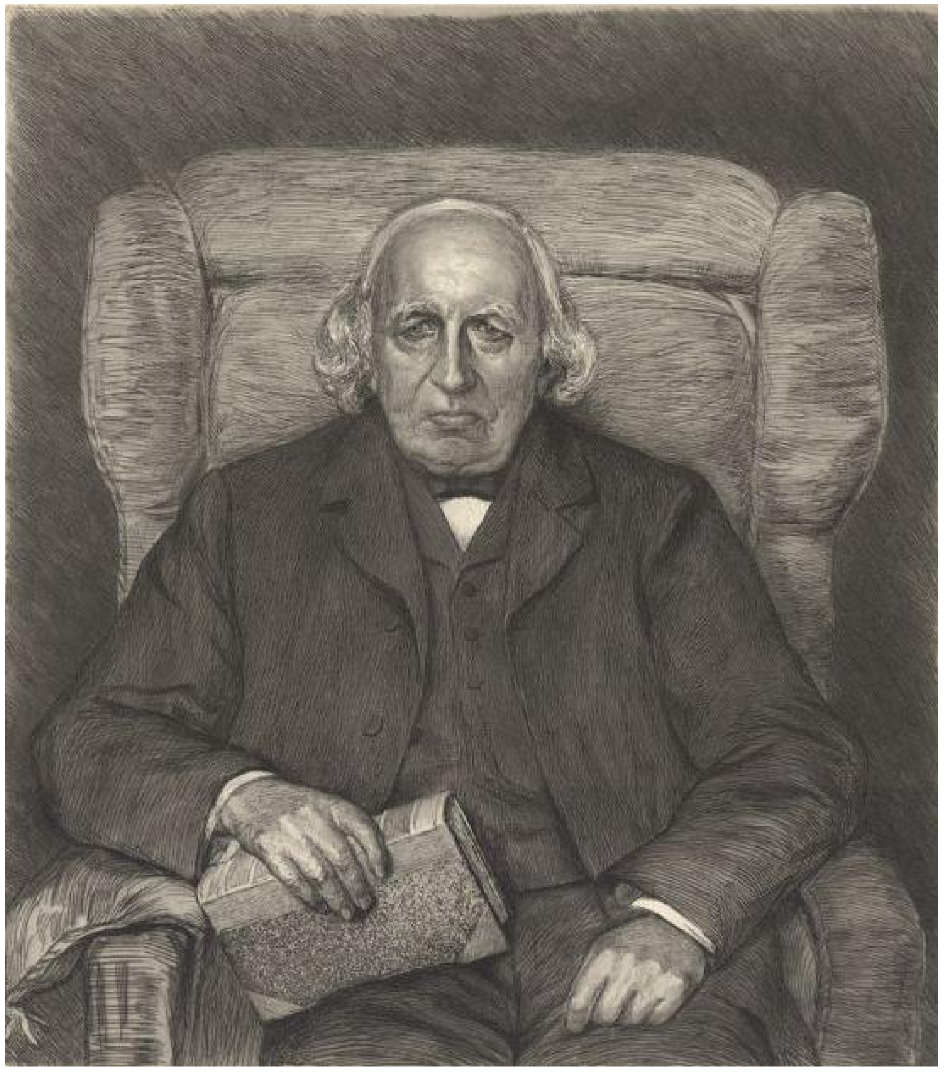
EXERCISE 2: Let Q be the principal part of a holomorphic function f on $U \subset \mathbb{C}^n$. Perform a linear coordinate change such that $Q(0, ..., 0, z_n) \neq 0$. **Prove that $\lim\limits_{z_n \to 0} \frac{F(0,z_n)}{z_n^k} \neq 0$.**

Weierstrass preparation theorem

DEFINITION: Let $z_1, ..., z_n$ be coordinate functions on $\mathbb{C}^n$. Denote the ring of germs of holomorphic functions on $\mathbb{C}^n$ depending on $z_1, ..., z_k$ by $\mathcal{O}_k$ **A Weierstrass polynomial** is a function $F \in \mathcal{O}_{n-1}[z_n]$, with the leading coefficient 1. In other words, $F = A_0 + z_n A_1 + ... + a_{k-1} z_n^{k-1} + z_n^k$, where A_i are germs of holomorphic functions on $\mathbb{C}^n$ depending on $z_1, ..., z_{n-1}$. **A Weierstrass polynomial is often written as $P(z, z_n)$,** where z denotes the collection $z_1, ..., z_{n-1}$.

THEOREM: (Weierstrass preparation theorem)
Let $F \in \mathcal{O}_n$ be a germ of a holomorphic function, with zero of order k in 0, such that $\lim_{z_n \to 0} \frac{F(0,z_n)}{z_n^k} \neq 0$ **Then in a certain neighbourhood of 0, F can be decomposed as $F = uP(z, z_n)$,** where $u \in \mathcal{O}_n$ is an invertible holomorphic function, and $P(z, z_n)$ a Weierstrass polynomial of degree k. Moreover, **such a decomposition is unique.**

Proof: Next lecture



Karl Theodor Wilhelm Weierstraß
(1815 - 1897)

Weierstrass preparation theorem: its applicability

EXERCISE 1: Let f be a holomorphic function on $U \subset \mathbb{C}^n$ which has in 0 a zero of order k. Prove that **for any coordinate system centered in 0, the limit $\lim\limits_{z_n \to 0} \frac{f(0,...,0,z_n)}{z_n^k}$ is finite.**

EXERCISE 2: Let Q be the principal part of a holomorphic function f on $U \subset \mathbb{C}^n$. Perform a linear coordinate change such that $Q(0,...,0,z_n) \neq 0$. **Prove that $\lim\limits_{z_n \to 0} \frac{F(0,z_n)}{z_n^k} \neq 0$.**

THEOREM: (Weierstrass preparation theorem)
Let $F \in \mathcal{O}_n$ be a germ of a holomorphic function, with zero of order k in 0, such that $\lim\limits_{z_n \to 0} \frac{F(0,z_n)}{z_n^k} \neq 0$ **Then in a certain neighbourhood of 0, F can be decomposed as $F = uP(z, z_n)$,** where $u \in \mathcal{O}_n$ is an invertible holomorphic function, and $P(z, z_n)$ a Weierstrass polynomial of degree k. Moreover, **such a decomposition is unique.**

REMARK: Exercises 1-2 imply that the Weierstrass preparation theorem can be applied to any holomorphic function f which satisfies $Q(0,...,0,z_n) \neq 0$. In particular, **it can be applied after a linear coordinate change $z_1,...,z_n \longrightarrow A(z_1),...,A(z_n)$ for $A \in GL(n)$ outside of a measure 0 set.**

COROLLARY: For any countable set of holomorphic functions $f_1, f_2, ...$, **there exists a coordinate system such that all f_i satisfy the assumptions of the Weierstrass preparation theorem.**