Complex analytic spaces

lecture 4: Weierstrass preparation theorem (2)

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August 16, 2023

Order of zero of a holomorphic function (reminder)

DEFINITION: Let f be a holomorphic function on $U \subset \mathbb{C}^n$, vanishing in $0 \in U$, and $f(z) = \sum_{i=0}^{\infty} P_i(t_1, ..., t_n)$ be its Taylor series, where P_i are homogeneous polynomials of degree i. We say that **f has zero of order k in 0**, or **zero of multiplicity k**, if $P_0 = ... = P_{k-1} = 0$.

DEFINITION: Let $f(t_1, ..., t_n)$ be a holomorphic function which has zero of order d in 0. The **principal part** f is a homogeneous polynomial $Q(t_1, ..., t_n)$ of degree d such that $f - Q$ has zero of order $\geqslant d + 1$.

DEFINITION: Let $z_1, ..., z_n$ be coordinate functions on $\mathbb{C}^n$. Denote the ring of germs of holomorphic functions on $\mathbb{C}^n$ depending on $z_1, ..., z_k$ by $\mathcal{O}_k$ **A Weierstrass polynomial** is a function $F \in \mathcal{O}_{n-1}[z_n]$, with the leading coefficient 1. In other words, $F = A_0 + z_n A_1 + ... + a_{k-1} z_n^{k-1} + z_n^k$, where A_i are germs of holomorphic functions on $\mathbb{C}^n$ depending on $z_1, ..., z_{n-1}$. **A Weierstrass polynomial is often written as $P(z, z_n)$,** where z denotes the collection $z_1, ..., z_{n-1}$.

Weierstrass preparation theorem (reminder)

EXERCISE 1: Let f be a holomorphic function on $U \subset \mathbb{C}^n$ which has in 0 a zero of order k. Prove that **for any coordinate system centered in 0, the limit $\lim\limits_{z_n \to 0} \frac{f(0,...,0,z_n)}{z_n^k}$ is finite.**

EXERCISE 2: Let Q be the principal part of a holomorphic function f on $U \subset \mathbb{C}^n$. Perform a linear coordinate change such that $Q(0,...,0,z_n) \neq 0$. **Prove that $\lim\limits_{z_n \to 0} \frac{F(0,z_n)}{z_n^k} \neq 0$.**

THEOREM: **(Weierstrass preparation theorem)**
Let $F \in \mathcal{O}_n$ be a germ of a holomorphic function, with zero of order k in 0, and $Q \in \mathbb{C}[z_1,...,z_n]$ its principal part. Assume that $Q(0,z_n) \neq 0$. **Then in a certain neighbourhood of 0, F can be decomposed as $F = uP(z,z_n)$,** where $u \in \mathcal{O}_n$ is an invertible holomorphic function, and $P(z,z_n)$ a Weierstrass polynomial of degree k. Moreover, **such a decomposition is unique.**

Proof: Later today

COROLLARY: For any countable set of holomorphic functions $f_1, f_2, ...$, **there exists a coordinate system such that all f_i satisfy the assumptions of the Weierstrass preparation theorem.** ■

Newton identity

DEFINITION: Let $\alpha_1, ..., \alpha_l$ be a collection of independent variables, and e_i the coefficients of a polynomials $t^l + e_{l-1}t^{l-1} + ... + e_1 t + e_0 := \prod_{i=1}^l (t + \alpha_i)$. The polynomials $e_i(\alpha_1, ..., \alpha_l)$ are called **the elementary symmetric polynomials on α_i.**

THEOREM: (Newton identity)
Let $Q_j := \sum_i \alpha^j$. **Then the elementary symmetric polynomials $e_0, ..., e_{l-1}$ are expressed through $Q_1, ..., Q_l$** with rational coefficients.

Proof: More precisely, **Newton identity** gives

$$ke_k = \sum_{i=0}^{k-1} (-1)^i e_{k-i} Q_i.$$

To see that, write the generating function $E(t) := \prod_i (1 - t\alpha_i) = \sum_i (-1)^i t^i e_i$. Taking a derivation in t, we obtain

$$\frac{E'(t)}{E(t)} = \sum_i \frac{-\alpha_i}{1 - \alpha_i t} = -\sum_i \sum_{j=1}^{\infty} \alpha_i^j t^{j-1}.$$

Let $Q(t) := \sum_{j=1}^{\infty} Q_j t^j$. The previous formula gives $tE' = -EQ$, which gives the Newton identity. ■

Zeroes of the logarithmic derivative

Exercise 3: Let f be a holomorphic function on a disc $\Delta \subset \mathbb{C}$, not vanishing on its boundary $\partial\Delta$, and $S_k(f) := \frac{1}{2\pi\sqrt{-1}} \int_{\partial\Delta} \frac{f'}{f} z^k dz$. **Then $S_k(f) = \sum \alpha_i^k$, where α_i is the set of zeroes of f,** taken with their multiplicities.

Hint: Use the Cauchy formula.

EXERCISE: Use this to prove **Rouché's theorem**: **for any a family f_t of holomorphic functions on Δ, continuously depending on $t \in \mathbb{R}$ and not vanishing on $\partial\Delta$,** the number of zeroes of f_t in Δ is constant in t.

Weierstrass preparation theorem: strategy of the proof

THEOREM: (Weierstrass preparation theorem)
Let $F \in \mathcal{O}_n$ be a germ of a holomorphic function, with zero of order k in 0, and $Q \in \mathbb{C}[z_1, ..., z_n]$ its principal part (the lowest degree homogeneous term of its Taylor series). Assume that $Q(0, z_n) \neq 0$. **Then in a certain neighbourhood of 0, F can be decomposed as $F = uP(z, z_n)$,** where $u \in \mathcal{O}_n$ is an invertible holomorphic function, and $P(z, z_n)$ a Weierstrass polynomial of degree k. Moreover, **such a decomposition is unique.**

To prove the Weierstrass preparation theorem, we consider the set of zeroes of F as an analytic subvariety of $\mathbb{C}^n$, projected to $\mathbb{C}^{n-1}$ as a k-fold ramified covering, **and construct a Weiertrass polynomial with the same zero set as F.**

Weierstrass preparation theorem: strategy of the proof (2)

This is done as follows. We consider the function $\tilde{F}_{z_1,...,z_{n-1}}(z_n) = F(z_1, ..., z_n)$ as a function on Δ, expressed as a power series with coefficients in $\mathcal{O}_{n-1}$. It is a holomorphic functions on Δ, smooth and non-vanishing on its boundary (we shrink Δ if necessary). Then we write the Newton polynomials of the zeroes of $\tilde{F}_{z_1,...,z_{n-1}}(z_n)$ in Δ, using the logarithmic derivative formula (Exercise 3), as holomorphic functions on $z_1, ..., z_{n-1}$. We use the Newton formula to express these Newton polynomials through the elementary symmetric polynomials. **This gives a finction $P(z, z_n)$, polynomial in z_n, and vanishing in the same points as $\tilde{F}_{z_1,...,z_{n-1}}(z_n)$ with the same multiplicities.**

EXERCISE: (Riemann removable singularity theorem)
Let $Z \subset U$ be a complex analytic subset, and f a locally bounded function on $U\backslash Z$, holomorphic on $U\backslash Z$. **Prove that f is extended to a holomorphic function on U.**

Weierstrass preparation theorem (proof)

The proof of Weierstrass preparation theorem:
Since $\frac{F(0,z_n)}{z_n^k} = Q(0,z_n) \neq 0$, there is a polydisc $\Delta(n-1,1) := B_r(z_1,...z_{n-1}) \times \Delta_{r'}(z_n)$ of biradius r, r', such that $F(z,z_n) \neq 0$ when $|z_n| = r'$. We write the decomposition $F = uP$ in this polydisc.

Step 1: Let $\mathfrak{S}_k(z) := S_k(F(z,\cdot))$, where $z \in B_r(z_1,...z_{n-1})$, and $S_k(f) := \frac{1}{2\pi\sqrt{-1}} \int_{\partial\Delta} \frac{f'}{f} z^k dz$. By Exercise 3, **$\mathfrak{S}_0(z)$ is equal to the number of zeroes of $F(z,\cdot)$ on the disc $\Delta_{r'}$.** Since $\mathfrak{S}_0(z)$ is continuous, the number of zeroes is constant (this is Rouché's theorem, by the way).

Step 2: Let $e_l(z)$ be the elementary polynomials of these zeroes, denoted as $\alpha_i(z)$. Exercise 3 gives $\sum \alpha_i^l(z) = \mathfrak{S}_l(z)$. **Using the Newton identity, we express $e_l(z)$ through $\mathfrak{S}_l(z)$, obtaining $e_l(z)$ as a holomorphic function on $z_1, ..., z_{n-1}$.**

Step 3: Let $P(z,z_n) := z_n^k + \sum_{i=0}^{k-1} (-1)^i e_i(z) z^i$. This function has the same zeroes as $F(z,z_n)$ with the same multiplicities. **Therefore, the fraction $u := \frac{F(z,z_n)}{P(z,z_n)}$ is nowhere vanishing in $\Delta(n-1,1)$;** this function is complex differentiable, hence it is holomorphic and invertible in $\Delta(n-1,1)$. We obtain $F = Pu$. ■