Complex analytic spaces

lecture 5: Weierstrass division theorem

Misha Verbitsky

IMPA, sala 236,

August 21, 2023, 13:30

Gauss lemma

EXERCISE: Let R be a ring without zero divisors. **Prove that the polynomial ring $R[t]$ has no zero divisors.**

THEOREM: ("Gauss lemma")
Let R be a factorial ring. **Then the ring of polynomials $R[t]$ is also factorial.**

Proof: See the next slide.

DEFINITION: An element $p \in R$ is called **irreducible**, if for any decomposition $p = rs$ in R, either r or s is invertible.

DEFINITION: Let R be a factorial ring. A polynomial $P(t) \in R[t]$ is called **primitive** if the greatest common divisor of its coefficients is 1.

Lemma 1: Let $P_1, P_2 \in R[t]$ be primitive polynomials. **Then their product is also primitive**.

Proof: Let $p \in R$ be an irreducible element. Since the polynomials P_1, P_2 are primitive, they are non-zero modulo p. Since the ring $R/(p)$ has no zero divisors, **the product P_1P_2 is non-zero in $R/(p)[t]$,** hence the greatest common divisor of the coefficients of P_1P_2 is not divisible by p. ■

Irreducibility of polynomials in $R[t]$ and $K[t]$

Lemma 1: Let $P_1, P_2 \in R[t]$ be primitive polynomials. **Then their product is also primitive**.

Lemma 2: Let R be a factorial ring, and K its fraction field. **Then any primitive polynomial $P \in R[t]$, which is irreducible in $R[t]$, is also irreducible in $K[t]$.**

Proof: Assume that P is decomposable in $K[t]$. Then $rP = P_1P_2$, where $P_1, P_2 \in R[t]$ and $r \in R$. Let s_1, s_2 be the greatest common divisors of the coefficients of P_1, P_2. Then $rP = s_1s_2P_1'P_2'$, and P_1', P_2' are primitive. In this case $P_1'P_2'$ is primitive (Lemma 1), hence the greatest common divisor of the coefficients of $s_1s_2P_1'P_2'$ is s_1s_2. Since P is also primitive, the greatest common divisor of the coefficients of $rP = s_1s_2P_1'P_2'$ is r. **Then $\frac{r}{s_1s_2}$ is invertible, and P is decomposable in $R[t]$.** ■

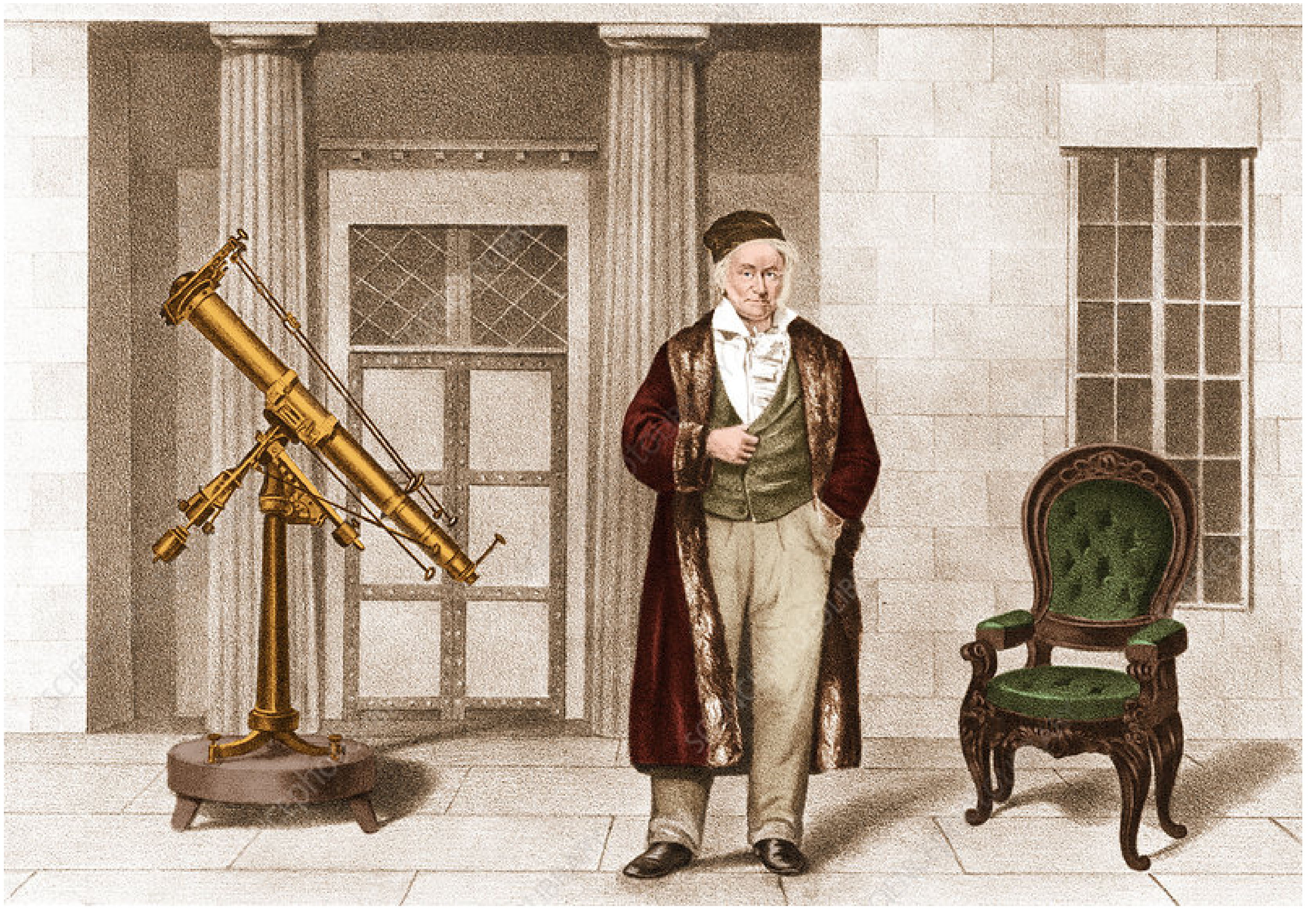
THEOREM: ("Gauss lemma")
Let R be a factorial ring. **Then the ring of polynomials $R[t]$ is also factorial.**

Proof: Let K be the fraction field of R. The ring $K[t]$ is factorial, because it is Euclidean. Lemma 2 implies that **an irreducible decomposition of a primitive polynomial $P(t) \in R[t]$ is uniquely determined by its prime decomposition in $K[t]$, hence it is unique.** A non-primitive polynomial is decomposed as a product of the greatest common divisor of its coefficients and a primitive polynomial, hence its prime decomposition is also unique. ∎

COROLLARY: The affine space $\mathbb{C}^n$ is a normal variety. Moreover, **for any variety X with factorial ring $\mathcal{O}_X$ of regular functions, the product $X \times \mathbb{C}^n$ is also normal.**

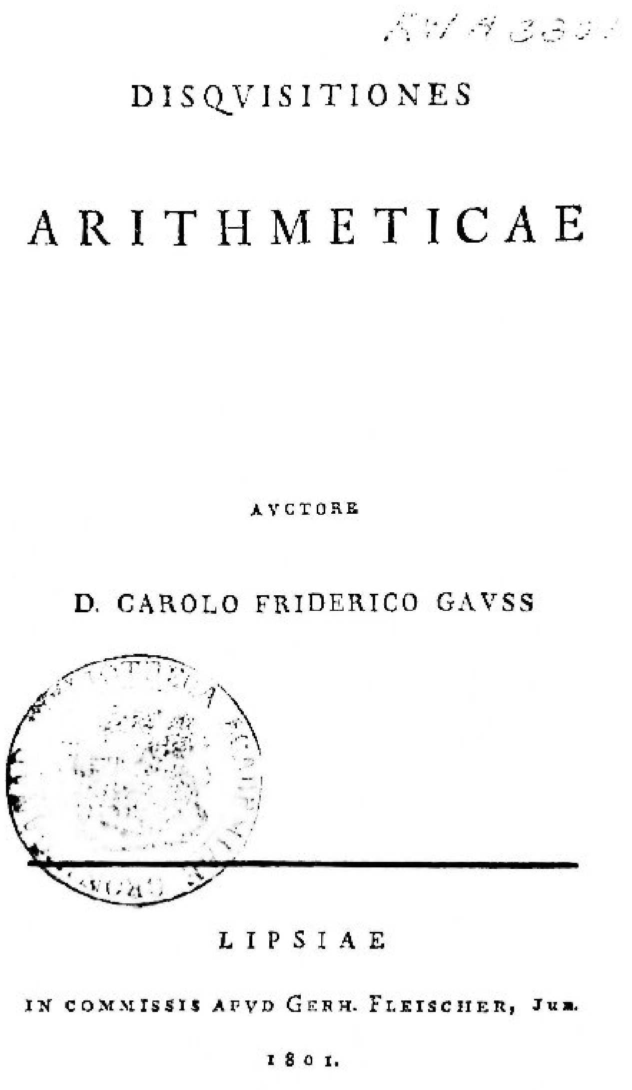
Proof: As we have shown previously, $\mathcal{O}_{X\times\mathbb{C}^n} = \mathcal{O}_X \otimes_{\mathbb{C}} \mathbb{C}[t_1, ..., t_n] = \mathcal{O}_X[t_1, ..., t_n]$. This ring is factorial by Gauss lemma. ∎

Carl Friedrich Gauss



Carl Friedrich Gauss (1777 - 1855)

Disquisitiones Arithmeticae


DISQVISITIONES

ARITHMETICAE

AVCTORE

D. CAROLO FRIDERICO GAVSS

LIPSIAE

IN COMMISSIS APVD GERH. FLEISCHER, IUN.

1801.

"Disquisitiones Arithmeticae",
written by Gauss in 1798, in Latin, when he was 21.
This book contains "Gauss Lemma".

Weierstrass preparation theorem (reminder)

DEFINITION: Let $z_1, ..., z_n$ be coordinate functions on $\mathbb{C}^n$. Denote the ring of germs of holomorphic functions on $\mathbb{C}^n$ depending on $z_1, ..., z_k$ by $\mathcal{O}_k$ **A Weierstrass polynomial** is a function $F \in \mathcal{O}_{n-1}[z_n]$, with the leading coefficient 1. In other words, $F = A_0 + z_n A_1 + ... + a_{k-1} z_n^{k-1} + z_n^k$, where A_i are germs of holomorphic functions on $\mathbb{C}^n$ depending on $z_1, ..., z_{n-1}$. **A Weierstrass polynomial is often written as $P(z, z_n)$,** where z denotes the collection $z_1, ..., z_{n-1}$.

THEOREM: (Weierstrass preparation theorem)
Let $F \in \mathcal{O}_n$ be a germ of a holomorphic function, with zero of order k in 0, and $Q \in \mathbb{C}[z_1, ..., z_n]$ its principal part. Assume that $Q(0, z_n) \neq 0$. **Then in a certain neighbourhood of 0, F can be decomposed as $F = uP(z, z_n)$,** where $u \in \mathcal{O}_n$ is an invertible holomorphic function, and $P(z, z_n)$ a Weierstrass polynomial of degree k. Moreover, **such a decomposition is unique.**

Proof: Lecture 4.

COROLLARY: For any countable set of holomorphic functions $f_1, f_2, ...$, **there exists a coordinate system such that all f_i satisfy the assumptions of the Weierstrass preparation theorem.** ■

Carl Ludwig Siegel (1896 - 1981)

The proof of WPT and of Weierstrass division theorem below comes from "Topics in complex function theory," by Carl L. Siegel.

Carl Ludwig Siegel, 1896 - 1981

The ring of germs of holomorphic functions is factorial

DEFINITION: A germ of holomorphic function $f \in \mathcal{O}_n$ is called **indecomposable** if for any decomposition $f = g_1 g_2, .., g_n$, all g_i except (at most) one are invertible.

PROPOSITION: Let $f \in \mathcal{O}_n$. **Then f can be decomposd as $f = f_1 ... f_r$, where all f_i are indecomposable.** Moreover, **such a decomposition is unique**, up to an order and an invertible multiplier.

Proof: It is sufficient to prove this when f is a Weierstrass polynomial. Decomposing f into a product of irreducible polynomials, we obtain a decomposition $f = f_1 ... f_r$. It remains to prove that it is unique.

Step 1: Using induction, **we may assume that $\mathcal{O}_{n-1}$ is factorial.** Gauss lemma implies that $\mathcal{O}_{n-1}[z_n]$ is also factorial.

Step 2: Let $g \in \mathcal{O}_n$ be an indecomposable germ dividing the product of indecomposable germs uv. Applying the WPT, we may assume that $g, u, v \in \mathcal{O}_n$ are Weierstrass polynomials in the same coordinate system; these polynomials are clearly indecomposable. **Then g divides uv in $\mathcal{O}_{n-1}[z_n]$.**

Step 3: By Step 1, $\mathcal{O}_{n-1}[z_n]$ is factorial, hence g **divides** u **or** v, which implies the uniqueness of prime decomposition on $\mathcal{O}_n$. ∎

Division with remainders

EXERCISE: Let $f, g \in \mathbb{C}[t]$ be polynomials, with all roots of g in the unit disc $\Delta \subset \mathbb{C}$. **Prove that the function**

$$h(z) = \frac{1}{2\pi\sqrt{-1}} \int_{\partial\Delta} \frac{f(\zeta)}{g(\zeta)} \frac{1}{\zeta - z} d\zeta. \quad (*)$$

is polynomial in z. Prove that $h(z)$ **is equal to the quotient f % g under division with remainder.**

Division with remainders (2)

Proposition 1: Let $f(t)$ be a holomorphic functions on the unit disc $\Delta\subset\mathbb{C}$, and $g(t)$ a polynomial which does not vanish on the boundary $\partial\Delta$. Then the function

$$h(z)=\frac{1}{2\pi\sqrt{-1}}\int_{\partial\Delta}\frac{f(\zeta)}{g(\zeta)}\frac{1}{\zeta-z}d\zeta.\quad (*)$$

is holomorphic in the disc Δ. Moreover, **the function $r(z):=f(z)-g(z)h(z)$ is a unital polynomial of degree smaller than $\deg g$.**

Proof. Step 1: The function $h(z)$ is clearly holomorphic, because (*) is decomposed in Taylor series in z, as usual. Indeed, **$v(z):=\int_{\partial\Delta}u(\zeta)\frac{1}{\zeta-z}d\zeta$ is holomorphic for any integrable function u on $\partial\Delta$.**

Step 2:

$$f(z)-h(z)g(z)=f(z)-\frac{1}{2\pi\sqrt{-1}}\int_{\partial\Delta}\left[g(z)\frac{f(\zeta)}{g(\zeta)}\frac{1}{\zeta-z}\right]d\zeta=$$

$$=\frac{1}{2\pi\sqrt{-1}}\int_{\partial\Delta}\left[\frac{f(\zeta)}{\zeta-z}-g(z)\frac{f(\zeta)}{g(\zeta)}\frac{1}{\zeta-z}\right]d\zeta=\frac{1}{2\pi\sqrt{-1}}\int_{\partial\Delta}\frac{f(\zeta)}{g(\zeta)}\cdot\frac{g(\zeta)-g(z)}{\zeta-z}d\zeta.$$

The function $\frac{g(\zeta)-g(z)}{\zeta-z}$ is a polynomial in z of degree $\deg g-1$. This implies that $r(z)=f(z)-h(z)g(z)\in\mathbb{C}[z]$, and $\deg r<\deg g-1$. ■

Weierstrass division theorem

As in the WPT, **we write $(z_1, ..., z_{n-1}, z_n)$ as (z, z_n).**

THEOREM: (Weierstrass division theorem)
Let $P(z, z_n) \in \mathcal{O}_{n-1}[z_n]$ be a Weierstrass polynomial of degree k, with $P(0, z_n) = z_n^k$. **Then every germ $F \in \mathcal{O}_n$ can be written as $F = hP + Q$,** where $Q(z, z_n)$ is a Weierstrass polynomial of degree $< k$.

Proof. Step 1: Since $\lim_{z_n \to 0} \frac{P(0,z_n)}{z_n^k} = 1$, in a certain polydisc $\Delta(n-1, 1) := B_r(z_1, ... z_{n-1}) \times \Delta_{r'}(z_n)$ we have $P(z, z_n) \neq 0$ when $|z_n| = r'$. We will construct the decomposition $F = hP + Q$ in this polydisc.

Step 2: Write

$$h(z, z_n) = \frac{1}{2\pi\sqrt{-1}} \int_{\partial\Delta} \frac{F(z, \zeta)}{P(z, \zeta)} \frac{1}{\zeta - z_n} d\zeta.$$

By Proposition 1, $Q := F - Ph$ is a polynomial in z_n of degree $< k$, with coefficients in $\mathcal{O}_{n-1}$ and the leading term 1, that is, a Weierstrass polynomial.

■