Complex analytic spaces

lecture 6: Germs of complex analytic varieties and Noetherianity

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Weierstrass preparation theorem (reminder)

DEFINITION: Let $z_1, ..., z_n$ be coordinate functions on $\mathbb{C}^n$. Denote the ring of germs of holomorphic functions on $\mathbb{C}^n$ depending on $z_1, ..., z_k$ by $\mathcal{O}_k$ **A Weierstrass polynomial** is a function $F \in \mathcal{O}_{n-1}[z_n]$, with the leading coefficient 1. In other words, $F = A_0 + z_n A_1 + ... + a_{k-1} z_n^{k-1} + z_n^k$, where A_i are germs of holomorphic functions on $\mathbb{C}^n$ depending on $z_1, ..., z_{n-1}$. **A Weierstrass polynomial is often written as $P(z, z_n)$,** where z denotes the collection $z_1, ..., z_{n-1}$.

THEOREM: (Weierstrass preparation theorem)
Let $F \in \mathcal{O}_n$ be a germ of a holomorphic function, with zero of order k in 0, and $Q \in \mathbb{C}[z_1, ..., z_n]$ its principal part. Assume that $Q(0, z_n) \neq 0$. **Then in a certain neighbourhood of 0, F can be decomposed as $F = uP(z, z_n)$,** where $u \in \mathcal{O}_n$ is an invertible holomorphic function, and $P(z, z_n)$ a Weierstrass polynomial of degree k. Moreover, **such a decomposition is unique.**

Proof: Lecture 4.

COROLLARY: For any countable set of holomorphic functions $f_1, f_2, ...$, **there exists a coordinate system such that all f_i satisfy the assumptions of the Weierstrass preparation theorem.** ■

Weierstrass division theorem (reminder)

THEOREM: (Weierstrass division theorem)
Let $P(z, z_n) \in \mathcal{O}_{n-1}[z_n]$ be a Weierstrass polynomial of degree k, with $P(0, z_n) = z_n^k$. **Then every germ $F \in \mathcal{O}_n$ can be written as $F = hP + Q$,** where $Q(z, z_n)$ is a Weierstrass polynomial of degree $< k$.

Proof. Step 1: Since $\lim_{z_n \to 0} \frac{P(0,z_n)}{z_n^k} = 1$, in a certain polydisc $\Delta(n-1, 1) := B_r(z_1, ... z_{n-1}) \times \Delta_{r'}(z_n)$ we have $P(z, z_n) \neq 0$ when $|z_n| = r'$. We will construct the decomposition $F = hP + Q$ in this polydisc.

Step 2: Write

$$h(z, z_n) = \frac{1}{2\pi\sqrt{-1}} \int_{\partial\Delta} \frac{F(z, \zeta)}{P(z, \zeta)} \frac{1}{\zeta - z_n} d\zeta.$$

By Proposition 1, $Q := F - Ph$ is a polynomial in z_n of degree $< k$, with coefficients in $\mathcal{O}_{n-1}$, that is, a Weierstrass polynomial. ■

Noetherian rings

DEFINITION: A ring R is called **Noetherian** if any increasing chain of ideals in R stabilizes: for any chain $I_1 \subset I_2 \subset I_3 \subset ...$ one has $I_n = I_{n+1} = I_{n+2} = ...$

EXERCISE: Prove that R is Noetherian if and only **if any ideal in R is finitely generated as an R-module.**

EXERCISE: Let R be a Noetherian ringm and M a finitely generated R-module. **Prove that any submodule $M_1 \subset M$ is also finitely generated.**

THEOREM: (Hilbert basis theorem)
Any finitely generated ring is Noetherian.

Proof: I will distribute a handout with the proof, if needed.

Ring of germs in Noetherian

THEOREM: (Emanuel Lasker, 1905)
The ring $\mathcal{O}_n$ of germs of holomorphic functions is Noetherian.
Proof. Step 1: Let $I \subset \mathcal{O}_n$ be an ideal, and $P \in I$ a non-zero element. By WPT, P is a Weierstrass polynomial, up to an invertible multiplier. Dividing P by an invertible multiplier if necessary, we can assume that $P(z, z_n)$ is a Weierstrass polynomial of degree k. Weierstrass division theorem implies that **the ring $\mathcal{O}_n/(P)$ is generated by $1, z_n, z_n^2, ..., z_n^{k-1}$ over $\mathcal{O}_{n-1}$.**

Step 2: This implies that $\mathcal{O}_n/(P)$ **is finitely generated as an $\mathcal{O}_{n-1}$-module.**

Step 3: Denote by π the natural homomorphism $\mathcal{O}_n \longrightarrow \mathcal{O}_n/(P)$. Using induction, we may assume that $\mathcal{O}_{n-1}$ is Noetherian. Then **the image $\pi(I) \subset \mathcal{O}_n/(P)$ is finitely generated as a $\mathcal{O}_{n-1}$-module.**

Step 4: Let $\xi_1, ..., \xi_N$ be generators of $\pi(I)$, and $\tilde{\xi}_1, ..., \tilde{\xi}_N$ their preimages in I. **Then $P, \tilde{\xi}_1, ..., \tilde{\xi}_N$ generate I as an $\mathcal{O}_n$-module.** Since any ideal in $\mathcal{O}_n$ is finitely generated, $\mathcal{O}_n$ is Noetherian. ■

COROLLARY: The ring $\mathcal{O}_{Z,x}$ of germs of holomorphic functions on a complex variety is Noetherian.
Proof: By definition, this ring is a quotient of $\mathcal{O}_n$. ■

Emanuel Lasker

Emanuel Lasker
(December 24, 1868 – January 11, 1941)

Irreducible complex varieties

DEFINITION: A complex variety Z is called **irreducible** if it cannot be decomposed as a union of complex subvarieties $Z = A_1 \cup A_2$ such that $A_1 \not\subset A_2$ and $A_2 \not\subset A_1$.

DEFINITION: **An irreducible component** of a complex variety Z is an irreducible subvariety $A \subset Z$ such that $Z = A \cup B$, for a complex subvariety $B \subset Z$ such that $A \not\subset B$ and $B \not\subset A$.

DEFINITION: **An irreducible decomposition** of a complex variety Z is a decomposition of Z into the union of its irreducible components.

REMARK: Just as in the category of algebraic varieties, **the irreducible decomposition of complex varieties always exists** (we prove this in the next lecture). However, there are two things which are different.

EXERCISE: Construct a connected complex variety **which has infinitely many irreducible components.**

EXERCISE: Construct an irreducible complex variety Z **and a connected open subset $U \subset Z$ such that U is not irreducible.**

Complex analytic subvarieties and their germs

DEFINITION: A complex analytic subset (or **a complex analytic subvariety**) of a complex manifold M is a closed subset $Z \subset M$ locally defined as the set of common zeros of a collection of holomorphic functions.

EXERCISE: Prove that **a finite union and any intersection of complex analytic subsets is a complex analytic subset.**

DEFINITION: Let $Z_1 \subset U_1, Z_2 \subset U_2$ be complex analytic subsets of open subsets $U_i \subset M$ containing $x \in M$. We say that **Z_1 and Z_2 have the same germ in x** if $Z_1 \cap U = Z_2 \cap U$ for some neighbourhood $U \subset U_1 \cap U_2$ containing x. Clearly, **this defines an equivalence relation. A germ of a complex analytic subset** in $x \in M$ is an equivalence class of complex-analytic subsets $Z \subset U$ under this relation.

EXERCISE: Prove that **a finite union and any intersection of germs of complex analytic subsets is a germ of a complex analytic subset.**

Irreducible germs of subvarieties

DEFINITION: Let A_1, A_2 be germs of complex analytic subsets in $x \in M$. **We say that $A_1 \subset A_2$** if $A_1 \cap U \subset A_2 \cap U$ for some open neighbourhood of $x \in M$.

DEFINITION: A germ Z of a complex analytic subset in $x \in M$ is called **irreducible** if for any decomposition $Z = A_1 \cup A_2$ into a union of two germs, we have $A_1 \subset A_2$ or $A_2 \subset A_2$. **An irreducible component** of a germ Z is an irreducible germ $Z_1 \subset Z$ such that $Z = Z_1 \cap Z_2$, for some other germ $Z_2 \not\supset Z_1$.

EXERCISE: Prove that **a germ of a smooth submanifold is irreducible.**

EXERCISE: Find an irreducible complex variety Z such that the germ of Z in x is not irreducible.

Germs and ideals

DEFINITION: Let Z be a germ of a complex analytic subset of M in x, and $f \in \mathcal{O}_{M,x}$ a germ of a function. We say that f **vanishes in** Z if there exists an open set $U \ni x$, a complex analytic subset $Z_u \subset U$ and a function $f_U \in \Gamma(\mathcal{O}_U)$ such that f is a germ of f_U, Z is a germ of Z_U, and $f_U\big|_{Z_U} = 0$.

DEFINITION: An ideal $I \subset R$ is called **radical** if R/I has no nilpotents.

EXERCISE: Let Z be a germ of a complex analytic subvariety in $x \in M$, and $\operatorname{Ann}(Z) \subset \mathcal{O}_{M,x}$ the ideal of all germs of holomorphic functions vanishing in Z. **Prove that $\operatorname{Ann}(Z)$ is a radical ideal.**

THEOREM: (Rückert)
"Complex-analytic Nulstellensatz"
Let Z be a germ of a complex analytic subvariety in $x \in M$, and $\operatorname{Ann}(Z) \subset \mathcal{O}_{M,x}$. Then Z is uniquely determined by $\operatorname{Ann}(Z)$. Moreover, **this correspondence defines a bijection between the set of germs of complex analytic subsets in $x \in M$ and the set of radical ideals in $\mathcal{O}_{M,x}$.**

Proof: Next week. ■

Irreducible subvarieties and prime ideals

PROPOSITION: A germ of subvariety $Z \subset M$ **is irreducible if and only if its ideal** $\operatorname{Ann}(Z) \subset \mathcal{O}_n$ **is prime.**

Proof. Step 1: Suppose that we have a non-trivial decomposition $Z = A \cup B$ onto a union of germs of subvarieties. Then there exist functions vanishing on B and not on A and functions vanishing on A and not on B. **The product of such functions belongs to** $\operatorname{Ann}(Z)$**, which is therefore not prime.**

Step 2: Conversely, assume that $\operatorname{Ann}(Z)$ is not prime. Then there exists $f, g \in \mathcal{O}_n$ which don't vanish on Z, such that $fg \in \operatorname{Ann}(Z)$. Let $A \subset Z$ be the zeros of f and $B \subset Z$ be the zeros of g; **then** $Z = A \cup B$**, hence** Z **is not irreducible.** ■