Complex analytic spaces

lecture 7: Irreducible decomposition and Noetherian topological spaces

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Noetherian rings (reminder)

DEFINITION: A ring R is called **Noetherian** if any ascending chain of ideals in R stabilizes: for any chain $I_1 \subset I_2 \subset I_3 \subset ...$ one has $I_n = I_{n+1} = I_{n+2} = ...$

EXERCISE: Prove that R is Noetherian if and only **if any ideal in R is finitely generated as an R-module.**

EXERCISE: Let R be a Noetherian ringm and M a finitely generated R-module. **Prove that any submodule $M_1 \subset M$ is also finitely generated.**

THEOREM: **(Hilbert basis theorem)**
Any finitely generated ring is Noetherian.

THEOREM: **(Emanuel Lasker, 1905)**
The ring $\mathcal{O}_n$ of germs of holomorphic functions is Noetherian.

COROLLARY: **The ring $\mathcal{O}_{Z,x}$ of germs of holomorphic functions on a complex variety is Noetherian.**

Proof: By definition, this ring is a quotient of $\mathcal{O}_n$. ■

Complex analytic subvarieties and their germs (reminder)

DEFINITION: A complex analytic subset (or **a complex analytic subvariety**) of a complex manifold M is a closed subset $Z \subset M$ locally defined as the set of common zeros of a collection of holomorphic functions.

EXERCISE: Prove that **a finite union and any intersection of complex analytic subsets is a complex analytic subset.**

DEFINITION: Let $Z_1 \subset U_1, Z_2 \subset U_2$ be complex analytic subsets of open subsets $U_i \subset M$ containing $x \in M$. We say that **Z_1 and Z_2 have the same germ in x** if $Z_1 \cap U = Z_2 \cap U$ for some neighbourhood $U \subset U_1 \cap U_2$ containing x. Clearly, **this defines an equivalence relation. A germ of a complex analytic subset** in $x \in M$ is an equivalence class of complex-analytic subsets $Z \subset U$ under this relation.

EXERCISE: Prove that **a finite union and any intersection of germs of complex analytic subsets is a germ of a complex analytic subset.**

Irreducible germs of subvarieties (reminder)

DEFINITION: Let A_1, A_2 be germs of complex analytic subsets in $x \in M$. **We say that $A_1 \subset A_2$** if $A_1 \cap U \subset A_2 \cap U$ for some open neighbourhood of $x \in M$.

DEFINITION: A germ Z of a complex analytic subset in $x \in M$ is called **irreducible** if for any decomposition $Z = A_1 \cup A_2$ into a union of two germs, we have $A_1 \subset A_2$ or $A_2 \subset A_2$. **An irreducible component** of a germ Z is an irreducible germ $Z_1 \subset Z$ such that $Z = Z_1 \cap Z_2$, for some other germ $Z_2 \not\supset Z_1$.

EXERCISE: Prove that **a germ of a smooth submanifold is irreducible.**

EXERCISE: Find an irreducible complex variety Z such that the germ of Z in x is not irreducible.

Existence of irreducible components in a germ

The following proposition is an immediate consequence of Noetherianity.

PROPOSITION: Any germ Z of a complex variety is a union of its irreducible components.

Proof: Assume otherwise; then for any non-trivial decomposition $Z = Z_1 \cup Z_2$ onto proper subvarieties $Z_i \ni x$, each of these subvarieties can be decomposed further, and so on. This gives a strictly decreasing sequence of germs $A_1 \supsetneq A_2 \supsetneq ... \supsetneq A_n \supsetneq ...$ of subvarieties in x. **Then the corresponding sequence of ideals** $\operatorname{Ann}(A_1) \subsetneq \operatorname{Ann}(A_2) \subsetneq ... \subsetneq \operatorname{Ann}(A_n) \subsetneq ...$ **in $\mathcal{O}_{Z,x}$ is strictly increasing, which contradicts Noetherianity.** ■

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Germs of irreducible subvarieties and prime ideals

PROPOSITION: A germ of subvariety $Z \subset M$ **is irreducible if and only if its ideal** $\operatorname{Ann}(Z) \subset \mathcal{O}_n$ **is prime.**

Proof. Step 1: Suppose that we have a non-trivial decomposition $Z = A \cup B$ onto a union of germs of subvarieties. Then there exist functions vanishing on B and not on A and functions vanishing on A and not on B. **The product of such functions belongs to** $\operatorname{Ann}(Z)$**, which is therefore not prime.**

Step 2: Conversely, assume that $\operatorname{Ann}(Z)$ is not prime. Then there exists $f, g \in \mathcal{O}_n$ which don't vanish on Z, such that $fg \in \operatorname{Ann}(Z)$. Let $A \subset Z$ be the zeros of f and $B \subset Z$ be the zeros of g; **then** $Z = A \cup B$**, hence** Z **is not irreducible.** ■

COROLLARY: A germ of subvariety $Z \subset M$ in $z \in M$ **is irreducible if and only if the ring** $\mathcal{O}_{Z,z}$ **has no zero divisors.** ■

Zariski topology

DEFINITION: Let R be a ring, and Spec(R) the set of prime ideals in R. Given an ideal $I \subset R$, let $Z_I \subset \operatorname{Spec}(R)$ be the set of all prime ideals containing I. Recall that a set $Z \subset \operatorname{Spec}(R)$ **is closed in Zariski topology** if $Z = Z_I$, for some ideal $I \subset R$.

CLAIM: A finite union and any intersection of Zariski closed sets is Zariski closed.

Proof. Step 1: Indeed, for any collection of ideals $\{I_\alpha\}$, a prime ideal $\mathfrak{p}$ contains all I_α if and only if it contains $I := \sum I_\alpha$, which gives $\bigcap_\alpha Z_{I_\alpha} = Z_I$.

Step 2: Also, for any two ideals $I, J \subset R$, a prime ideal $\mathfrak{p}$ contains IJ if it contains I or J. This gives $Z_I \cup Z_J \supset Z_{IJ}$. The converse implication is also true, which is proven by contradiction. Suppose that $\mathfrak{p} \not\supset I$ and $\mathfrak{p} \not\supset J$, but $\mathfrak{p} \supset IJ$. Then there exists $a \in I \backslash \mathfrak{p}$ and $b \in J \backslash \mathfrak{p}$. Since $\mathfrak{p}$ is prime, **this implies $ab \notin \mathfrak{p}$, which gives a contradiction because $ab \in IJ \subset \mathfrak{p}$.** ■

DEFINITION: This defines **the Zariski topology** on Spec(R), which is called **the Zariski spectrum** of R.

EXERCISE: Prove that any affine complex algebraic variety **is compact with respect to the Zariski topology.**

Noetherian topological spaces

DEFINITION: A topological space M is called **Noetherian** if any descending chain $M_0 \supset M_1 \supset M_2 \supset ...$ of closed subsets of M terminates.

EXAMPLE: Let R be a Noetherian ring, and Spec(R) its spectrum with the Zariski topology. **Then Spec(R) is Noetherian**, because descending chains of closed subsets correspond to ascending chains of ideals, and such chains always stabilize.

EXAMPLE: **An affine complex algebraic variety** is a subset $Z \subset \mathbb{C}^n$ defined by a family of polynomial equations. A subset $Z_1 \subset Z$ is called **Zariski closed** if it is also a variety. Clearly, this defines a topology on Z, also called **the Zariski topology**. Since each $Z_1 \subset Z$ is a zero set of an ideal, and any ascending chain of ideals stabilizes, **the Zariski topology on an affine complex algebraic variety is Noetherian.**

EXERCISE: Let M be an infinite topological space which admits a metric. **Prove that it is never Noetherian.**

The complex analytic Zariski topology

EXERCISE: Let M be a complex manifold. Define **the complex analytic Zariski topology** on M as a topology in which closed sets are complex analytic subsets of M. **Prove that it is not Noetherian when M is not compact.**

EXERCISE: Prove that M **is compact with respect to the usual topology if and only if M is compact with respect to the analytic Zariski topology.**

REMARK: When M is compact, this topology is Noetherian (we shall prove this today).

Irreducible topological spaces

DEFINITION: Let M be a topological space. We say that M is **irreducible** if for any decomposition $M = X \cup Y$ onto a union of two closed subsets, either $X \subset Y$ or $Y \subset X$.

CLAIM: Let M be a complex analytic variety, considered as a topological space with Zariski analytic topology. Assume that M is connected and its germs have no zero divisors. **Then M is irreducible.**

Proof: Consider a decomposition $M = X \cap Y$ as above, where X, Y are closed. Since M is connected, the sets X, Y intersects non-trivially. Consider a point $z \in X \cap Y$ such that in any open neighbourhood $U \ni z$, neither $X \cap U \subset Y \cap U$ nor $Y \cap U \subset X \cap U$ (such a point exists because both conditions are open, and M is connected). In this point, **the germ of M is not irreducible, hence $\mathcal{O}_{M,z}$ has zero divisors.** ■

CLAIM: Let $U \subset M$ be an open subset in a topological space, and Z an irreducible component of U. **Then its closure $\overline{Z}$ in M is irreducible.**

Proof: Suppose that $\overline{Z} = X_1 \cup X_2$ be a decomposition of $\overline{Z}$ onto a union of two closed subsets. Then $Z = (X_1 \cap U) \cup (X_2 \cap U)$ is a decomposition of Z onto closed subsets, hence either $X_i \cap U$ is equal to Z, for $i = 1$ or $i = 2$. Then $\overline{Z}$ is the closure of $X_i \cap U$, giving $X_i = Z$. ■

Irreducibility of Spec(R)

EXERCISE: Prove that $\bigcap_{\mathfrak{p}\in \operatorname{Spec}(R)} \mathfrak{p} = \operatorname{Nil}(R)$, where $\operatorname{Nil}(R)$ is the nilradical.

CLAIM: The space Spec(R) is irreducible **if and only if for any zero divisors $f, g \in R$, either f or g are nilpotent.**

Proof: Let R be a ring without zero divisors, and $\operatorname{Spec} R = Z_I \cup Z_J$ a non-trivial decomposition of R onto a union of Zariski closed subsets. Since $Z_I \subsetneq \operatorname{Spec} R$ and $Z_J \subsetneq \operatorname{Spec} R$, neither I nor J are contained in the nilradical; otherwise we would have $Z_I = \operatorname{Spec}(R)$ or $Z_J = \operatorname{Spec}(R)$. Since $Z_{IJ} = \operatorname{Spec}(R)$, we have $IJ \subset \bigcap_{\mathfrak{p}\in \operatorname{Spec}(R)} \mathfrak{p} = \operatorname{Nil}(R)$. **Therefore, R contains zero divisors which are not nilpotent.**

Conversely, if R contains non-nilpotent zero divisors f, g, with $fg = 0$, we have $\operatorname{Spec} R = Z_{(f)} \cup Z_{(g)}$, hence $\operatorname{Spec} R$ is not irreducible. ■

Irreducible components of a Noetherian topological spaces

DEFINITION: Let M be a topological space, and $Z \subset M$ a closed subset. We say that Z **is an irreducible component of M** if Z is irreducible and there exists a decomposition $M = Z \cup Z_1$, where Z_1 is closed in M and $Z_1 \not\supset Z$.

PROPOSITION: Let M be a Noetherian topological space. **Then M is a union of its irreducible components, which are finitely many.**

Proof: Let $\mathfrak{S}$ be the set of all decompositions $M = X_1 \cup X_2 \cup ... \cup X_n$ such that all X_i are closed and for all i one has $X_i \not\subset \bigcup_{j\neq i} X_j$. This set is partially ordered, with $S_2 \prec S_1$ if S_2 is a subdivision of S_1. **Since M is Noetherian, any chain of subdivisions $S_1 \succ S_2 \succ S_3 \succ ...$ stabilizes.** This implies that there exists a minimal subdivision, which is precisely a subdivision of M onto its irreducible components. ■

EXAMPLE: A complex algebraic variety $Z \subset \mathbb{C}^n$ is called **irreducible** if for any decomposition $Z = A \cup B$ onto a union of complex subvarieties, either $Z = A$ or $Z = B$. Since the Zariski topology on complex algebraic varieties is Noetherian, **every complex algebraic variety admits an irreducible decomposition.**

EXERCISE: Prove that a complex algebraic variety $Z \subset \mathbb{C}^n$ **is irreducible if and only if the ring of polynomial functions on Z has no zero divisors.**

Locally Noetherian topological spaces

DEFINITION: A topological space M is called **locally Noetherian** if for any descending chain of closed subsets $Z_1 \supset Z_2 \supset ...$ and any $m \in M$, there is a neighbourhood $U \ni m$ such that the chain $Z_1 \cap U \supset Z_2 \cap U \supset ...$ stabilizes.

CLAIM: Let M be a complex analytic variety equipped with the analytic Zariski topology. **Then M is locally Noetherian.**

Proof: Let $m \in M$ and $Z_1 \supset Z_2 \supset ...$ be a descending chain of complex subvarieties. Since any descending chain of germs of complex subvarieties stabilizes, there is an open (in the usual topology) neighbourhood $W \ni m$ such that $Z_1 \cap W \supset Z_2 \cap W \supset ...$ stabilizes in $Z_n \cap W$. **To prove that M is locally Noetherian it remains to show that W can be chosen analytic Zariski open.**

Step 2: The intersection $Z := \bigcap Z_i$ (which is complex analytic) coincides with Z_n in W. Consider the set A of all $z \in Z$ such that $Z_n \neq Z$ in a neighbourhood of z. We are going to prove that A is complex analytic. **Then $Z_1 \cap U \supset Z_2 \cap U \supset ...$ stabilizes for $U = M \backslash A$ which is Zariski open.**

Step 3: In a neighbourhood of $z_1 \in Z$, **the set A is obtained as a union of all irreducible components of Z_n which don't belong to Z,** hence it is complex analytic. ■

Irreducible decomposition in locally Noetherian topological spaces

THEOREM: Let M be a locally Noetherian topological space. **Then $M = \bigcup Z_i$, where Z_i are its irreducible components.** Moreover, **the decomposition $M = \bigcup Z_i$ is locally finite.**

Proof. Step 1: Let $\mathfrak{S}$ be the set of all locally finite decompositions $M = X_1 \cup X_2 \cup ... \cup X_n \cup ...$ such that all X_i are closed and for all i one has $X_i \not\subset \bigcup_{j\neq i} X_j$. This set is partially ordered, with $S_2 \prec S_1$ if S_2 is a subdivision of S_1. We are going to use Zorn lemma to show that **there exists a minimal element $M = Z_1 \cup Z_2 \cup ... \cup Z_n \cup ...$ in this partially ordered set.** Clearly, $M = Z_1 \cup Z_2 \cup ... \cup Z_n \cup ...$ is an irreducible decomposition.

Step 2: Since M is Noetherian, any chain of subdivisions $S_1 \succ S_2 \succ S_3 \succ ...$ stabilizes locally. Given an infinite chain of decompositions $S_1 \succ S_2 \succ S_3 \succ ....$ with S_i given by $M = X_1(i) \cup X_2(i) \cup ... \cup X_n(i) \cup ...$, let S be a decomposition of M onto the intersections of any infinite subset of $X_i(j)$. Since the chain S_i stabilizes locally and all S_i are locally finite, **the decomposition S is also locally finite.** On the other hand, $S \prec S_i$ for all i, hence the conditions for Zorn lemma are satisfied and $\mathfrak{S}$ contains a minimal element. ■

COROLLARY: Any complex analytic variety M is a union of its irreducible components.
Proof: Indeed, M is locally Noetherian in analytic Zariski topology. ■

Irreducible decomposition in compact complex analytic variety

COROLLARY: Let M be a compact complex analytic space, and $M = Z_1 \cup Z_2 \cup ... \cup Z_n \cup ...$ its irreducible decomposition. **Then the set $\{Z_i\}$ of its irreducible components is finite.**

Proof: Since any analytic Zariski open set is open, M is compact in analytic Zariski topology as well. Locally in analytic Zariski topology, M has finitely many irreducible components. Passing to a finite subcover, we obtain that M is a finite union of open subsets U_i with each U_i having only finitely many irreducible components. Analytic Zariski closure of an irreducible set is irreducible. Therefore the irreducible components of M are analytic Zariski closures of the irreducible components of U_i, and there are only finitely many of them. ■