Complex analytic spaces

lecture 8: Regular coordinate system

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Weierstrass preparation and division theorems (reminder)

DEFINITION: Let $z_1, ..., z_n$ be coordinate functions on $\mathbb{C}^n$. Denote the ring of germs of holomorphic functions on $\mathbb{C}^n$ depending on $z_1, ..., z_k$ by $\mathcal{O}_k$ **A Weierstrass polynomial** is a function $F \in \mathcal{O}_{n-1}[z_n]$, with the leading coefficient 1. In other words, $F = A_0 + z_n A_1 + ... + a_{k-1} z_n^{k-1} + z_n^k$, where A_i are germs of holomorphic functions on $\mathbb{C}^n$ depending on $z_1, ..., z_{n-1}$. **A Weierstrass polynomial is often written as $P(z, z_n)$,** where z denotes the collection $z_1, ..., z_{n-1}$.

THEOREM: **(Weierstrass preparation theorem)**
Let $F \in \mathcal{O}_n$ be a germ of a holomorphic function, with zero of order k in 0, and $Q \in \mathbb{C}[z_1, ..., z_n]$ its principal part. Assume that $Q(0, z_n) = z_n^k$. **Then in a certain neighbourhood of 0, F can be decomposed as $F = uP(z, z_n)$,** where $u \in \mathcal{O}_n$ is an invertible holomorphic function, and $P(z, z_n)$ a Weierstrass polynomial of degree k. Moreover, **such a decomposition is unique.**

THEOREM: **(Weierstrass division theorem)**
Let $P(z, z_n) \in \mathcal{O}_{n-1}[z_n]$ be a Weierstrass polynomial of degree k, with $P(0, z_n) = z_n^k$. **Then every germ $F \in \mathcal{O}_n$ can be written as $F = hP + Q$,** where $Q(z, z_n) \in$ is a polynomial of degree $< k$.

Complex analytic subvarieties and their germs (reminder)

DEFINITION: A complex analytic subset (or **a complex analytic subvariety**) of a complex manifold M is a closed subset $Z \subset M$ locally defined as the set of common zeros of a collection of holomorphic functions.

EXERCISE: Prove that **a finite union and any intersection of complex analytic subsets is a complex analytic subset.**

DEFINITION: Let $Z_1 \subset U_1, Z_2 \subset U_2$ be complex analytic subsets of open subsets $U_i \subset M$ containing $x \in M$. We say that **Z_1 and Z_2 have the same germ in x** if $Z_1 \cap U = Z_2 \cap U$ for some neighbourhood $U \subset U_1 \cap U_2$ containing x. Clearly, **this defines an equivalence relation. A germ of a complex analytic subset** in $x \in M$ is an equivalence class of complex-analytic subsets $Z \subset U$ under this relation.

EXERCISE: Prove that **a finite union and any intersection of germs of complex analytic subsets is a germ of a complex analytic subset.**

Irreducible germs of subvarieties (reminder)

DEFINITION: Let A_1, A_2 be germs of complex analytic subsets in $x \in M$. **We say that $A_1 \subset A_2$** if $A_1 \cap U \subset A_2 \cap U$ for some open neighbourhood of $x \in M$.

DEFINITION: A germ Z of a complex analytic subset in $x \in M$ is called **irreducible** if for any decomposition $Z = A_1 \cup A_2$ into a union of two germs, we have $A_1 \subset A_2$ or $A_2 \subset A_2$. **An irreducible component** of a germ Z is an irreducible germ $Z_1 \subset Z$ such that $Z = Z_1 \cap Z_2$, for some other germ $Z_2 \not\supset Z_1$.

EXERCISE: Prove that **a germ of a smooth submanifold is irreducible.**

EXERCISE: Find an irreducible complex variety Z such that the germ of Z in x is not irreducible.

PROPOSITION: Any germ Z of a complex variety is a union of its irreducible components.

COROLLARY: Let $Z \subset M$ be a germ of subvariety in $z \in M$. **Then the following are equivalent:** (a) Z is irreducible (b) $\mathcal{O}_{Z,z}$ has no zero divisors (c) the ideal $J_z \subset \mathcal{O}_{M,z}$ is prime.

Noether's normalization lemma

Today we are going to prove a complex-analytic version of the following theorem.

(Noether's normalization lemma)
Let $X \subset \mathbb{C}^n$ be an irreducible algebraic variety, $z_1, ..., z_n$ coordinates on $\mathbb{C}^n$. Assume that $z_1, ..., z_k$ is a transcendence basis in the fraction field $k(X)$. Then **there exists a linear coordinate change $z_i' := z_i + \sum_{j=1}^{n-k} \lambda_{j+k} z_{j+k}$, such that the projection $\Pi_k : X \longrightarrow C^k$ to the first k arguments is a finite, dominant morphism.**

We are going to show that any irreducible germ of a complex variety admits a holomorphic map φ to a hypersurface $X \subset \mathbb{C}^{k+1}$, and φ induces an invertible function on the corresponding fraction fields. Moreover, the coordinate projection $\pi : X \longrightarrow \mathbb{C}^k$ is proper, and every point has finitely many preimages. Finally, outside of a proper subvariety $D \subset \mathbb{C}^k$, called **the discriminant** the map π is a finite covering.

This implies, for instance, that **for any germ of complex subvariety $X \subset \mathbb{C}^n$ there exists a proper subvariety $Z \subset X$ and a hypersurface $D \subset \mathbb{C}^d$ together with a holomorphic covering $X \backslash Z \longrightarrow \mathbb{C}^d \backslash D$.**

Regular coordinate systems for an ideal

Consider the ring of germs of functions on $\mathbb{C}^n$ depending only on the first d coordinates. We identify this ring with $\mathcal{O}_d \subset \mathcal{O}_n$.

THEOREM: Let J be an ideal in $\mathcal{O}_n$. There exists a coordinate system $z_1, ..., z_d, z_{d+1}, ..., z_n$ in a neighbourhood of 0 such that

1. $J_d = 0$, **where** $J_d := \mathcal{O}_d \cap J$.
2. The ideal J **is generated by a collection of Weierstrass polynomials in** $\mathcal{O}_{i-1}[z_i]$, $i = d+1, ..., n$.

Proof. Step 1: Let $P_1, ..., P_N$ be generators of J. Choose a coordinate system where all these generators are Weierstrass polynomials, $P_i \in \mathcal{O}_{n-1}[z_n]$. Then J **is generated by** $P_i(z_n)$ **and the intersection** $J_{n-1} := J \cap \mathcal{O}_{n-1}$.

Step 2: Applying induction in n, we may assume that the theorem is already proven for J_{n-1}. **Then** J **is generated by** $P_i(z_n)$ **and the generators of** J_{n-1}. ■

DEFINITION: In this situation, $z_1, ..., z_n$ is called **a regular coordinate system for** J.

Regular coordinate systems: geometric interpretation

THEOREM: Let J be an ideal in $\mathcal{O}_n$. There exists a coordinate system $z_1, ..., z_d, z_{d+1}, ..., z_n$ ("a regular coordinate system for J") in a neighbourhood of 0 such that

1. **$J_d = 0$, where $J_d := \mathcal{O}_d \cap J$.**
2. The ideal J **is generated by a collection of Weierstrass polynomials in $\mathcal{O}_{i-1}[z_i]$, $i = d+1, ..., n$.**

REMARK: Let $J \subset \mathcal{O}_n$ be an ideal of functions vanishing in a germ of a complex analytic variety Z. Then the first condition is equivalent to the following. Let $\Pi_d : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be the projection to first d coordinates. **The condition (1) means that $\Pi_d(Z)$ is not contained in any proper analytic subset $Z' \subset \mathbb{C}^d$.**

REMARK: In this situation, the second condition is an algebraic restatement of the following geometric observation. Consider the projection $\Pi_d : Z \longrightarrow \mathbb{C}^d$ to the first d coordinates. **Then the preimage of every point is finite.**

Artin's primitive element theorem

Exercise 1: Let $[K:k]$ be a finite field extension, $\operatorname{char} k = 0$. **Prove that there exist only finitely many intermediate subfields $K \supset K_1 \supset k$.**

Exercise 2: Let k be an infinite field, W a vector space over K, and S a union of finitely many subspaces $V_i \subset W$ of positive codimension. **Prove that $W \backslash S$ is infinite.**

DEFINITION: Let $[K:k]$ be a finite field extension. An element $x \in K$ is called **primitive** if it generates K over k.

THEOREM: (Artin's primitive element theorem)
Let $[K:k]$ be a finite field extension, $\operatorname{char} k = 0$. **Then there exists a primitive element $x \in K$.**

Proof: Take for x an element which does not belong to intermediate subfields $K \supsetneq K' \supset k$. Such an element exists by Exercise 2, because k is infinite, and K' belongs to a finite set of subspaces of positive codimension (Exercise 1). **Then x is primitive, because it generates a subfield which is equal to K.** ■

Artin's primitive element theorem (the second version)

We will use a stronger version of this theorem

THEOREM: (Artin's primitive element theorem, the second version) Let $[K:k]$ be a finite field extension, $\operatorname{char} k = 0$, and $x_1, ..., x_n \in K$ a collection of multiplicative generators. Then there exists an infinite subset $U \subset k^n$ such that **for any point $(\lambda_1, ..., \lambda_n) \in k^n$, the element $u := \sum \lambda_i x_i$ is primitive**.

Proof: Let $\{K_j \subsetneq K\}$ be the set of all intermediate subfields $K \supsetneq K' \supset k$. Exercise 1 implies that there are only finitely many. Consider the vector space W generated by $x_1, ..., x_n$ over k. For each j, the intersection $K_j \cap W$ is a subspace of positive codimension, because W multiplicatively generates K. **Exercise 2 implies that $U := W \backslash \bigcup K_j$ is infinite.** ∎

Regular coordinate systems: finite exensions

THEOREM: Let J be an ideal in $\mathcal{O}_n$. There exists a coordinate system $z_1, ..., z_d, z_{d+1}, ..., z_n$ in a neighbourhood of 0 such that 1. $J_d = 0$**, where** $J_d := \mathcal{O}_d \cap J$**.** 2. The ideal J **is generated by a collection of Weierstrass polynomials** $P_i(z_i) \in \mathcal{O}_{i-1}[z_i]$**,** $i = d+1, ..., n$**.**

Lemma 1: Let $P_{d+1}, ..., P_n$ be the Weierstrass polynomials, constructed above, $\deg P_i = s_i$. Then $\mathcal{O}_n/J$, as a $\mathcal{O}_d$-module, is generated by monomials on $z_{d+1}, ..., z_n$ of degree less than s_i on each variable z_i, $i = d+1, ..., n$.

Proof. Step 1: Using induction in n, we may assume that **lemma is proven for any function** $F \in \mathcal{O}_{n-1}$**.**

Step 2: Let $F \in \mathcal{O}_n$. Using Weierstrass division theorem, we may assume that $F = fP_n + Q$, where $Q(z_n) \in \mathcal{O}_{n-1}[z_n]$, $\deg Q < s_n$. Using Step 1, we express the coefficients of $Q(z_n)$ in terms of monomials of degree less than s_i on each variable z_i, $i = d+1, ..., n-1$. This is used to express Q as a linear combination of monomials on $z_{d+1}, ..., z_n$. Since $F = Q \mod J$, any element of $\mathcal{O}_n$ can be thus expressed. ■

COROLLARY: Assume that J is a prime ideal. **Lemma 1 implies that** $\mathcal{O}_n/J$ **is a finite extension of** $\mathcal{O}_d$**.**

Finiteness theorem

COROLLARY: (Finiteness theorem)
Let $z_1, ..., z_n$ be a regular coordinate system for an ideal $J \subset \mathcal{O}_n$, and $\mathcal{O}_d$ holomorphic functions, depending only on $z_1, ..., z_d$. Then **the ring $\mathcal{O}_n/J$ is finitely generated as a $\mathcal{O}_d$-module.**
Proof: It is generated by a finite number of coordinate monomials. ■

THEOREM: (Primitive element theorem)
Let $J \subset \mathcal{O}_n$ be a prime ideal, such that $\mathcal{O}_n/J$ is finitely generated as a $\mathcal{O}_d$-module. **Then for an open, dense subset U in the vector space $\langle z_{d+1}, ..., z_n \rangle$, the function $u = \sum_{i=d+1}^n \lambda_i z_i$ generates the fraction field $k(\mathcal{O}_n/J)$ over $k(\mathcal{O}_d)$.**
Proof: Follows from Artin's primitive element theorem, because $z_{d+1}, ..., z_n$ multiplicatively generate $\mathcal{O}_n/J$. ■

DEFINITION: **A germ of a hypersurface**, or **a germ of divisor** in $\mathbb{C}^n$ is a germ of subvariety given by a single holomorphic equation. The coordinate projection Π_d from $\mathbb{C}^n$ to $\mathbb{C}^{n-1}$, taking $(z_1, ..., z_n)$ to $(z_1, ..., z_{n-1})$ **is finite on the germ of hypersurface $Z \subset \mathbb{C}^n$** if Z is a common zero set of a Weierstrass polynomial $P(z_n) \in \mathcal{O}_{n-1}(z_n)$.

EXERCISE: Prove that for such a morphism, **the number of preimages $\#(\Pi_d^{-1}(z))$ is constant, if counted with multiplicities.**

A germ of a subvariety mapped to a hypersurface

THEOREM: Let J be a prime ideal in $\mathcal{O}_n$, and $z_1, ..., z_d, ..., z_n$ a regular coordinate system for J. Assume that $u := \sum_{i=d+1}^n \lambda_i z_i$ generates the fraction field $k(\mathcal{O}_n/J)$ over $k(\mathcal{O}_d)$. Define a map $\mathfrak{u}: \mathbb{C}^n \longrightarrow \mathbb{C}^{d+1}$, $\mathfrak{u}(z_1, ..., z_n) = (z_1, ..., z_d, u)$.

(1) **The map $\mathfrak{u}$ defines a holomorphic map $\varphi: Z \longrightarrow Z_u$ from the germ Z of common zeroes of J to a germ of a hypersurface $Z_u \subset \mathbb{C}^{d+1}$.**

(2) **The projection $\Pi_d: Z_u \longrightarrow \mathbb{C}^d$ to the first d coordinates is finite.**

(3) The map $Z \xrightarrow{\mathfrak{u}} Z_u$ **induces an isomorphism $k(\mathcal{O}_{d+1}/(P_u)) \xrightarrow{\sim} k(\mathcal{O}_n/J)$ on the fraction fields.**

Proof. Step 1: Take a regular coordinate system $(z_1, ..., z_d, ..., z_n$. Consider a primitive element $u = \sum_{i=d+1}^n \lambda_i z_i$, generating $k(\mathcal{O}_n/J)$ over $k(\mathcal{O}_d)$, and let $\mathcal{P}_u(t) \in \mathcal{O}_d[t]$ be its minimal polynomial. **Since $\mathcal{O}_n/J$ is finitely generated over $\mathcal{O}_d$, the polynomial $\mathcal{P}_u(t)$ has coefficients in $\mathcal{O}_d$ and leading term 1.** Let $Z_u \subset \mathbb{C}^{d+1}$ be the zero set of $\mathcal{P}_u(t)$ in $(z_1, ..., z_d, t)$.

Step 2: The map $\mathfrak{u}: \mathbb{C}^n \longrightarrow \mathbb{C}^{d+1}$, $(z_1, ..., z_n) \xrightarrow{\mathfrak{u}} \left(z_1, ..., z_d, u = \sum_{i=d+1}^n \lambda_i z_i\right)$ **takes Z to Z_u.** Indeed, if all elements of J vanish in $(z_1, ..., z_n)$, the function $\mathcal{P}_u(u) \in J$ also vanishes on $(z_1, ..., z_n)$. This map is finite, because $\mathcal{P}_u(t)$ is a Weierstrass polynomial.

Step 3: The isomorphism $k(\mathcal{O}_n/J) = k(\mathcal{O}_d[t]/(P_u(t)))$ **follows from the definition of a primitive element.** ■