Complex analytic spaces

lecture 10: Rückert's Nullstellensatz

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September 6, 2023, 17:00

Regular coordinate systems (reminder)

Consider the ring of germs of functions on $\mathbb{C}^n$ depending only on the first d coordinates. We identify this ring with $\mathcal{O}_d \subset \mathcal{O}_n$.

THEOREM: Let J be an ideal in $\mathcal{O}_n$. There exists a coordinate system $z_1, ..., z_d, z_{d+1}, ..., z_n$ in a neighbourhood of 0 such that

1. $J_d = 0$, **where** $J_d := \mathcal{O}_d \cap J$.
2. The ideal J **is generated by a collection of Weierstrass polynomials in** $\mathcal{O}_{i-1}[z_i]$, $i = d+1, ..., n$.

DEFINITION: In this situation, $z_1, ..., z_n$ is called **a regular coordinate system for** J.

REMARK: Let $J \subset \mathcal{O}_n$ be an ideal of functions vanishing in a germ of a complex analytic variety Z. Then the first condition is equivalent to the following. Let $P_d : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be the projection to first d coordinates. **The condition (1) means that $\Pi_d(Z)$ is not contained in any proper analytic subset $Z' \subset \mathbb{C}^d$.**

REMARK: In this situation, the second condition is an algebraic restatement of the following geometric observation. Consider the projection $\Pi_d : Z \longrightarrow \mathbb{C}^d$ to the first d coordinates. **Then the preimage of every point is finite.**

Finiteness theorem and primitive element theorem (reminder)

COROLLARY: (Finiteness theorem)
Let $z_1, ..., z_n$ be a regular coordinate system for an ideal $J \subset \mathcal{O}_n$, and $\mathcal{O}_d$ holomorphic functions, depending only on $z_1, ..., z_d$. Then **the ring $\mathcal{O}_n/J$ is finitely generated as a $\mathcal{O}_d$-module.**

Proof: It is generated by a finite number of coordinate monomials. ■

THEOREM: (Primitive element theorem)
Let J be a prime ideal in $\mathcal{O}_n$, and $z_1, ..., z_d, ..., z_n$ a regular coordinate system for J. Assume that $u := \sum_{i=d+1}^n \lambda_i z_i$ generates the fraction field $k(\mathcal{O}_n/J)$ over $k(\mathcal{O}_d)$. Define a map $\mathfrak{u}: \mathbb{C}^n \longrightarrow \mathbb{C}^{d+1}$, $\mathfrak{u}(z_1, ..., z_n) = (z_1, ..., z_d, u)$.

(1) **The map $\mathfrak{u}$ defines a holomorphic map $\varphi: Z \longrightarrow Z_u$ from the germ Z of common zeroes of J to a germ of a hypersurface $Z_u \subset \mathbb{C}^{d+1}$.**

(2) **The projection $\Pi_d: Z_u \longrightarrow \mathbb{C}^d$ to the first d coordinates is finite.**

(3) The map $Z \xrightarrow{\mathfrak{u}} Z_u$ **induces an isomorphism $k(\mathcal{O}_{d+1}/(P_u)) \xrightarrow{\sim} k(\mathcal{O}_n/J)$ on the fraction fields.**

The image of projection in regular coordinates

Proposition 1: Let $J \subset \mathcal{O}_n$ be a prime ideal, $(z_1, ..., z_d, ... z_n)$ a regular coordinate system, and $\Pi_d : Z \longrightarrow \mathbb{C}^d$ be the projection to the first d coordinates. **Then the image of Π_d does not lie in a proper analytic subset of $\mathbb{C}^d$.**

Proof. Step 1: Using the primitive element theorem, we find $u = \sum_{i=d+1}^n \lambda_i z_i$ which generates the fraction field $k(\mathcal{O}_n/J)$ over $k(\mathcal{O}_d)$. Let $\mathcal{P}_u(t) \in \mathcal{O}_d[t]$ be its minimal polynomial. Denote by $Z_u \subset (z_1, ..., z_d, t)$ a hypersurface defined by the equation $\mathcal{P}_u(t) = 0$. The projection $Z \longrightarrow \mathcal{O}_{d+1}$ taking $(z_1, ..., z_n)$ to $(z_1, ..., z_d, u)$ maps Z to Z_u and induces an isomorphism of fraction fields $k(\mathcal{O}_{Z_u}) \longrightarrow k(\mathcal{O}_Z)$, hence **the image of Z in Z_u does not lie in a proper analytic subvatiety.**

Step 2: Since $\mathcal{P}_u(t)$ is a Weierstrass polynomial, **the projection of its zero set Z_u to $\mathbb{C}^d$ is surjective in a neighbourhood of zero.** Therefore, $\Pi_d(Z)$ does not lie in a proper analytic subset of $\mathbb{C}^d$. ■

Rückert Nullstellensatz for prime ideals

THEOREM: Let $J \subset \mathcal{O}_n$ be a prime ideal, Z its zero set, considered as a germ of a complex variety, and $J_Z \subset \mathcal{O}_n$ the set of germs of functions vanishing in Z. **Then $J_Z = J$.**

Proof. Step 1: Clearly, $J_Z \supset J$, and we need only to prove the inverse inclusion. By the finiteness theorem, $\mathcal{O}_n/J$ is finitely generated as a $\mathcal{O}_d$-module. Therefore, for each $f \in \mathcal{O}_n/J$, **f satisfies an equation $Q(f) = 0$, where $Q(t) = t^n + a_{n-1}t^{n-1} + ... + a_0 \in \mathcal{O}_d[t]$ is a polynomial in $\mathcal{O}_d[t]$.**

Step 2: Since $\mathcal{O}_n/J$ has no zero divizors, for any decomposition $Q(t) = Q_1(t)Q_2(t)$, either $Q_1(f) = 0$ or $Q_2(f) = 0$. Therefore, **we can assume that $Q(t) \in \mathcal{O}_d[t]$ is an irreducible polynomial.**

Step 3: Let $f \in J_Z/J$. Since f vanishes on Z, and $Q(f) = 0$, the term $a_0 \in \mathcal{O}_d$ also vanishes on 0. By Proposition 1, this implies that $a_0 = 0$. **This is impossible, because $Q(t)$ is irreducible.** ■

Rückert Nullstellensatz for radical ideals

DEFINITION: Let $J \subset R$ be an ideal. Define **the radical** $\sqrt{J}$ of J as intersection of all prime ideal which contain J. We say that J is **a radical ideal** if $J = \sqrt{J}$.

Exercise 1: Prove that $a \in \sqrt{J}$ **if and only if** $a^n \in J$ **for some** $n > 0$**.** Equivalently, **prove that the nilradical of the ring** A/J **is the intersection of all prime ideals.**

THEOREM: (**Rückert Nullstellensatz**)
Let $J \subset \mathcal{O}_n$ be an ideal, Z its zero set, considered as a germ of a complex variety, and $J_Z \subset \mathcal{O}_n$ the set of germs of functions vanishing in Z. **Then** $J_Z = \sqrt{J}$**.**

Proof. Step 1: Since $\sqrt{J} = \{a \in \mathcal{O}_n \mid a^N \in J\}$ (Exercise 1), the zero sets of J and $\sqrt{J}$ are the same: $Z_J = Z_{\sqrt{J}}$.

Step 2: Let $\mathfrak{P}$ be the set of prime ideals containing J. Applying Step 1 and $\sqrt{J} = \bigcap_{J' \in \mathfrak{P}} J'$, we obtain $Z_J = Z_{\sqrt{J}} = \bigcup_{J' \in \mathfrak{P}} Z_{J'}$,

Step 3: Clearly, $J_Z \supset J$. For any $f \in J_Z$, the function f vanishes on $Z_{J'}$ for any $J' \in \mathfrak{P}$. By Rückert Nullstellensatz for prime ideals, $J_{Z_{J'}} = J'$, hence f belongs to $\bigcap_{J' \in \mathfrak{P}} J' = \sqrt{J}$. ■

Walther Rückert (1907-1984)

Walther Rückert, "Zum Eliminationsproblem der Potenzreihenideale", Math. Ann. 107 (1932), p. 259-281.

Verabschiedung des Oberschulamtspräsidenten Hermann Silber und Amtseinführung des neuen Präsidenten Dr. Rückert. 11. April 1964: Kultusminister Gerhard Storz (Mitte) beim Händedruck mit Dr. Rückert (rechts).

Complex Analysis in the Golden Fifties: R. Remmert

...At bottom of the arguments needed for the local theory of complex spaces is the WEIERSTRASS Preparation Theorem (1860). This theorem marks the beginning of the algebraization of the foundations of local function theory. The next important contribution towards algebraization was made in 1905 by E. LASKER (world champion of chess from 1894 till 1924). He showed that all rings $\mathcal{O}_n$ are noetherian and factorial. However this paper remained unknown. Even W. RÜCKERT, a student of KRULL, does not refer to it in his now classical Math. Annalen-paper from 1931 "Zum Eliminationsproblem der Potenzreihenideale". RÜCKERT'S proofs for noetherian and factorial are the classroom proofs of today. RUCKERT proudly writes:

"In dieser Arbeit wird gezeigt, daß eine sachgemäßte Behandlung nur formale Methoden, also keine funktionentheoretischen Hilfsmittel benötigt (In this paper we show that an appropriate treatment only needs formal methods and no function theoretic devices)."

RÜCKERT'S paper also contains the analytic HILBERT Nullstellensatz. The importance of RÜCKERT'S work was not recognized in his time, the paper fell into limbo. It was more then twenty years later that complex analysts slowly became algebraically minded [W. RÜCKERT, 1906 - 1984, his father was Minister in Baden till 1933; from 1964 - 1970 W. RÜCKERT was Präsident of the Oberschulamt Nordbaden].