Complex analytic spaces

lecture 11: Discriminant

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September 11, 2023, 13:30

Regular coordinate systems (reminder)

Consider the ring of germs of functions on $\mathbb{C}^n$ depending only on the first d coordinates. We identify this ring with $\mathcal{O}_d \subset \mathcal{O}_n$.

THEOREM: Let J be an ideal in $\mathcal{O}_n$. There exists a coordinate system $z_1, ..., z_d, z_{d+1}, ..., z_n$ in a neighbourhood of 0 such that

1. $J_d = 0$, **where** $J_d := \mathcal{O}_d \cap J$.
2. The ideal J **is generated by a collection of Weierstrass polynomials in** $\mathcal{O}_{i-1}[z_i]$, $i = d+1, ..., n$.

DEFINITION: In this situation, $z_1, ..., z_n$ is called **a regular coordinate system for** J.

REMARK: Let $J \subset \mathcal{O}_n$ be an ideal of functions vanishing in a germ of a complex analytic variety Z. Then the first condition is equivalent to the following. Let $P_d : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be the projection to first d coordinates. **The condition (1) means that $\Pi_d(Z)$ is not contained in any proper analytic subset $Z' \subset \mathbb{C}^d$.**

REMARK: In this situation, the second condition is an algebraic restatement of the following geometric observation. Consider the projection $\Pi_d : Z \longrightarrow \mathbb{C}^d$ to the first d coordinates. **Then the preimage of every point is finite.**

Finiteness theorem and primitive element theorem (reminder)

COROLLARY: (Finiteness theorem)
Let $z_1, ..., z_n$ be a regular coordinate system for an ideal $J \subset \mathcal{O}_n$, and $\mathcal{O}_d$ holomorphic functions, depending only on $z_1, ..., z_d$. Then **the ring $\mathcal{O}_n/J$ is finitely generated as a $\mathcal{O}_d$-module.**

Proof: It is generated by a finite number of coordinate monomials. ∎

THEOREM: (Primitive element theorem)
Let J be a prime ideal in $\mathcal{O}_n$, and $z_1, ..., z_d, ..., z_n$ a regular coordinate system for J. Assume that $u := \sum_{i=d+1}^n \lambda_i z_i$ generates the fraction field $k(\mathcal{O}_n/J)$ over $k(\mathcal{O}_d)$. Define a map $\mathfrak{u}: \mathbb{C}^n \longrightarrow \mathbb{C}^{d+1}$, $\mathfrak{u}(z_1, ..., z_n) = (z_1, ..., z_d, u)$.

(1) **The map $\mathfrak{u}$ defines a holomorphic map $\varphi: Z \longrightarrow Z_u$ from the germ Z of common zeroes of J to a germ of a hypersurface $Z_u \subset \mathbb{C}^{d+1}$.**

(2) **The projection $\Pi_d: Z_u \longrightarrow \mathbb{C}^d$ to the first d coordinates is finite.**

(3) The map $Z \xrightarrow{\mathfrak{u}} Z_u$ **induces an isomorphism $k(\mathcal{O}_{d+1}/(P_u)) \xrightarrow{\sim} k(\mathcal{O}_n/J)$ on the fraction fields.**

Banach fixed point theorem

LEMMA: (Banach fixed point theorem/ "contraction principle")
Let $U \subset \mathbb{R}^n$ be a closed subset, and $f: U \longrightarrow U$ a map which satisfies $|f(x) - f(y)| < k|x - y|$, where $k < 1$ is a real number (such a map is called **"contraction"**). **Then f has a fixed point, which is unique.**

Proof. Step 1: Uniqueness is clear because for two fixed points x_1 and x_2 $|f(x_1) - f(x_2)| = |x_1 - x_2| < k|x_1 - x_2|$.

Step 2: Existence follows because the sequence $x_0 = x, x_1 = f(x), x_2 = f(f(x)), ...$ satisfies $|x_i - x_{i+1}| \leqslant k|x_{i-1} - x_i|$ which gives $|x_n - x_{n+1}| < k^n a$, where $a = |x - f(x)|$. Then $|x_n - x_{n+m}| < \sum_{i=0}^m k^{n+i} a \leqslant k^n \frac{1}{1-k} a$, hence $\{x_i\}$ is a Cauchy sequence, and converges to a limit y, which is unique.

Step 3: $f(y)$ is a limit of a sequence $f(x_0), f(x_1), ... f(x_i), ...$ which gives $y = f(y)$. ■

EXERCISE: Find a counterexample to this statement when U is open and not closed.

Inverse function theorem

THEOREM: Let $U, V \subset \mathbb{C}^n$ be open subsets, and $f: U \longrightarrow V$ a holomorphic map. Suppose that the differential of f is everywhere invertible. **Then f is locally a diffeomorphism.**

Proof. Step 1: Let $x \in U$. Without restricting generality, we may assume that $x = 0$, $U = B_r(0)$ is an open ball of radius r, and **in U one has $\frac{|f(x_1) - \varphi(x_1)|}{|x - x_1|} < 1/2$.** Replacing f with $-f \circ (D_0 f)^{-1}$, where $D_0 f$ is differential of f in 0, **we may assume also that $D_0 f = -\operatorname{Id}$.**

Step 2: In these assumptions, $|f(x) + x| < 1/2|x|$ in a sufficiently small neighbourhood, hence $\psi_s(x) := f(x) + x - s$ is a contraction. This map maps $\overline{B}_{r/2}(0)$ to itself when $s < r/4$. By Banach fixed point theorem, **$\psi_s(x) = x$ has a unique fixed point x_s, which is obtained as a solution of the equation $f(x) + x - s = x$, or, equivalently, $f(x) = s$.** Denote the map $s \longrightarrow x_s$ by g.

Step 3: By construction, $fg = \operatorname{Id}$. Applying the chain rule, we find that g is also differentiable. ■

REMARK: This proof works for real manifolds just as well as for complex manifolds.

Smooth points of a subvariety

DEFINITION: Let $Z \subset \mathbb{C}^n$ be a complex analytic subset. A point $z \in Z$ is called **smooth** if for an open neighbourhood U of z in $\mathbb{C}^n$, the intersection $U \cap Z$ is a smooth submanifold. A point which is not smooth is called **singular.**

CLAIM: A complex analytic subset $Z \subset \mathbb{C}^n$ **is smooth as a real submanifold if and only if it is smooth as a complex submanifold.**

Proof: Suppose that $Z \subset \mathbb{C}^n$ is a complex subvariety which is smooth as a real submanifold. Consider the projection $\varphi: \mathbb{C}^n \longrightarrow \mathbb{C}^k$ which induces an isomorphism $d\varphi: T_z Z \longrightarrow \mathbb{C}^k$. Such a projection exists because Z is smooth. Then $\varphi: Z \longrightarrow \mathbb{C}^k$ is a complex analytic diffeomorphism in a neighbourhood of z, by (a holomorphic version of) inverse function theorem. ■

REMARK: Today we shall prove that **the set of smooth point on any complex variety is Zariski open and dense**.

Discriminant of a Weierstrass polynomial

REMARK: Let $P(t) = t^n + a_{n-1}t^{n-1} + ... + a_0$ be a polynomial, α_i its roots, and $Q(\alpha_1, ..., \alpha_n)$ a symmetric polynomial on α_i. Since any symmetric polynomial can be expressed polynomially through elementary symmetric polynomial, **$Q(\alpha_1, ..., \alpha_n)$ can be expressed polynomially through the coefficients a_i.**

DEFINITION: Let $P(t) = \prod_i (t-\alpha_i) = t^n + a_{n-1}t^{n-1} + ... + a_0$ be a polynomial. **The discriminant** D_P of P is the product $\prod_{i<j}(\alpha_i - \alpha_j)^2$ expressed as a polynomial on the coefficients of $P(t)$: $\prod_{i<j}(\alpha_i - \alpha_j)^2 = D_P(a_{n-1}, a_{n-2}, ..., a_0)$.

REMARK: Let $P(z_n) \in \mathcal{O}_{n-1}[z_n]$ be a Weierstrass polynomial, and D_P its discriminant. Since D_P is polynomially expressed through the coefficients of P, and coefficients belong to $\mathcal{O}_{n-1}$, **we can (and will) consider the discriminant as a function on $\mathbb{C}^{n-1}$.**

Weierstrass polynomial, its derivative and its discriminant

Lemma 1: Let $P(z_n) \in \mathcal{O}_{n-1}[z_n]$ be a Weierstrass polynomial, $P(0,0,...,0,z_n) = z^k$, and D_P its discriminant, considered an element in $\mathcal{O}_{n-1}$. Assume that $D_p = 0$. **Then the polynomials $P(z_n), P'(z_n) \in \mathcal{O}_{n-1}[z_n]$ are not coprime**.

Proof. Step 1: Since $\lim_{z_n \to 0} \frac{P(0,z_n)}{z_n^k} = 1$, in a certain polydisc $\Delta(n-1,1) := B_r(z_1, ... z_{n-1}) \times \Delta_{r'}(z_n)$ we have $P(z, z_n) \neq 0$ when $|z_n| = r'$. Fix $z \in B_r(z_1, ... z_{n-1})$ Then the function $t \to P(z,t)$ has precisely k zeros $\alpha_1, ..., \alpha_k$ on a disc $\Delta_{r'}$, and $D_P(z) = \prod_{i<j}(\alpha_i - \alpha_j)^2$. **If $D_P = 0$ on the disc $B_r(z_1, ... z_{n-1})$, the polynomial $P(z,t)$ is not coprime with $\frac{d}{dt}P(z,t)$ because $P(z,t)$, considered as a function of t, has multiple roots.**

Step 2: Consider the polynomials $P(z_n), P'(z_n) \in \mathcal{O}_{n-1}[z_n]$. Using the Euclidean algorithm, we can express their largest common denominator as $Q(z_n) = A(z_n)P(z_n) + B(z_n)P'(z_n)$, where $A, B \in \mathcal{O}_{n-1}[z_n]$. If P, P' are coprime, $Q \in \mathcal{O}_{n-1}$ is a non-zero germ of a function independent from z_n. Take $z \in \mathbb{C}^{n-1}$ such that $Q(z) \neq 0$. **In this point, $D_P(z) = \prod_{i<j}(\alpha_i - \alpha_j)^2 \neq 0$, hence the $D_P \neq 0$ when $P(z_n), P'(z_n)$ are coprime.** ■

Discriminants and coverings

Theorem 1: Let $P(z_n) \in \mathcal{O}_{n-1}[z_n]$ be a Weierstrass polynomial, $P(0,0,...,0,z_n) = z^k$, $X \subset \mathbb{C}^n$ its zero set and D_P its discriminant, considered an element in $\mathcal{O}_{n-1}$. We assume that $D_P \neq 0$. Denote by Z_D the zero set of D_P, and Z_X its preimage under the projection $\Pi: X \longrightarrow \mathbb{C}^{n-1}$ to the first $n-1$ coordinates. **Then $X \backslash Z_x$ is smooth and the projection $X \backslash Z_X \stackrel{\Pi}{\longrightarrow} \mathbb{C}^{n-1} \backslash Z_D$ is a non-ramified k-sheeted covering**.

DEFINITION: The sets Z_D and Z_X are called **the discriminant sets of** Π. The above theorem can be stated as **"Π is a non-ramified k-sheeted covering outside of its discriminant set."**

Proof. Step 1: Since $\lim_{z_n \to 0} \frac{P(0,z_n)}{z_n^k} = 1$, in a certain polydisc $\Delta(n-1,1) := B_r(z_1, ... z_{n-1}) \times \Delta_{r'}(z_n)$ we have $P(z, z_n) \neq 0$ when $|z_n| = r'$. In $\Delta(n-1,1)$, the projection $X \stackrel{\Pi}{\longrightarrow} \mathbb{C}^{n-1}$ takes precisely k points to one, by Rouché theorem, if counted with multiplicities. The discriminant is the set of all z where $z_n \to P(z, z_n)$ has roots with multiplicities, hence $X \backslash Z_X \stackrel{\Pi}{\longrightarrow} \mathbb{C}^{n-1} \backslash Z_D$ restricted to $\Delta(n-1,1)$ takes precisely k points to one. **It remains to prove that $X \backslash Z_X$ is smooth and $X \backslash Z_X \stackrel{\Pi}{\longrightarrow} \mathbb{C}^{n-1} \backslash Z_D$ a local diffeomorphism.**

Discriminants and coverings (2)

Proof. Step 1: Since $\lim_{z_n \to 0} \frac{P(0,z_n)}{z_n^k} = 1$, in a certain polydisc $\Delta(n-1,1) := B_r(z_1, ... z_{n-1}) \times \Delta_{r'}(z_n)$ we have $P(z, z_n) \neq 0$ when $|z_n| = r'$. In $\Delta(n-1,1)$, the projection $X \xrightarrow{\Pi} \mathbb{C}^{n-1}$ takes precisely k points to one, by Rouché theorem, if counted with multiplicities. The discriminant is the set of all z where $z_n \to P(z, z_n)$ has roots with multiplicities, hence $X \backslash Z_x \xrightarrow{\Pi} \mathbb{C}^{n-1} \backslash Z_D$ restricted to $\Delta(n-1,1)$ takes precisely k points to one. **It remains to prove that $X \backslash Z_x$ is smooth and $X \backslash Z_x \xrightarrow{\Pi} \mathbb{C}^{n-1} \backslash Z_D$ is a local diffeomorphism.**

Step 2: By the inverse function theorem, the zero set of a function F on $\mathbb{C}^n$ is smooth at every point where dF is non-zero. However, the 1-form dP evaluated on $\frac{d}{dz_n}$ if equal to $P'(z_n)$, **and it is non-zero on a point $(z, z_n) \in X \backslash Z_x$.**

Step 3: To prove that $X \backslash Z_x \xrightarrow{\Pi} \mathbb{C}^{n-1} \backslash Z_D$ is a local diffeomorphism, **it remains to show that the differential of the projection $X \backslash Z_x \xrightarrow{\Pi} \mathbb{C}^{n-1} \backslash Z_D$ is an isomorphism.** The tangent space to X in any $(z, z_n) \in X \backslash Z_x$ is equal to $\ker dP$. Clearly, $\ker dP$ does not intersect the space $(0, 0, ... 0, *) = \ker d\Pi$ in any point of $X \backslash Z_x$ because $dP(\frac{d}{dz_n}) = P'(z_n) \neq 0$. ■

Discriminant of the minimal polynomial

LEMMA: Let $Z \subset \mathbb{C}^n$ be a germ of an irreducible complex variety, $z_1, ..., z_d, z_{d+1}, ..., z_n$ the regular coordinates, $u = \sum_{i=d+1}^n \lambda_i z_i$ the primitive element which generates the fraction field $k(\mathcal{O}_n/J)$ over $k(\mathcal{O}_d)$, and $\mathcal{P}_u(t) \in \mathcal{O}_d[t]$ its minimal polynomial. Consider its discriminant D_{P_u} as a holomorphic function on $\mathbb{C}^d$. **Then $D_{P_u} \neq 0$.**

Proof: If $D_{P_u} = 0$, this means that $P_u(t)$ and $P_u'(t)$ are not coprime (Lemma 1). Then $P_u(t)$ is not irreducible, which is impossible, because $P_u(t)$ is a minimal polynomial, and $\frac{k(\mathcal{O}_d)[t]}{(P_u(t))}$ has no zero divisors. ■

COROLLARY: In these assumptions, let Z_u be the zero set of $P_u(t)$. Then, **in a sufficiently small neighbourhood of 0, the projection $Z_u \longrightarrow \mathbb{C}^d$ is a non-ramified k-sheeted covering outside of the discriminant set**.
Proof: Theorem 1. ■