Complex analytic spaces

lecture 12: Smooth points and bimeromorphic maps

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Pole divisor

DEFINITION: Let Z be a germ of a variety, f a non-zero holomorphic function on Z, and D_f its zero set. Assume that f is not identically zero on any of the irreducible components of Z. Then D_f is called **the zero divisor of f**.

DEFINITION: **A meromorphic function** on a complex analytic variety Z is a function, defined outside of a closed, nowhere Zariski dense complex analytic subset $D \subset Z$, and locally equal to a fraction $\frac{f}{g}$, where f, g are holomorphic functions, and $D_g = D$.

DEFINITION: Let f, g be germs of functions on a germ of complex variety Z. We say that f, g are **coprime** if they have no common divizors, vanishing in x. Let h be a meromorphic function on a germ Z of a variety, represented as $\frac{f}{g}$, where f, g are coprime. **The pole divisor** of h is the zero divisor of g.

CLAIM: Let h be a meromorphic function on a complex analytic variety Z. **Then there exists a subvariety $P_h \subset Z$ such that locally on each germ, P_h is the pole divisor of Z.**

Proof: Left as an exercise. ■

DEFINITION: In these assumptions, P_h is called **the pole divisor of Z**.

Meromorphic maps

DEFINITION: Meromorphic map $\varphi : Z \dashrightarrow Z_1$ of complex analytic varieties is a map, defined outside of a nowhere dense subvariety, which in local coordinates can be expressed by meromorphic functions.

EXAMPLE: Consider the map $\varphi : Z \longrightarrow Z_u$ constructed in the primitive element theorem. Since φ induces an isomorphism of the fraction field, **the inverse map φ^{-1} is meromorphic: the coordinate functions $z_{d+1}, ..., z_n$ on Z belong to the fraction field of the ring $\mathcal{O}_d[u] = \mathcal{O}_{Z_u}$,** hence **there exists a meromorphic map $Z_u \dashrightarrow Z$ taking $(z_1, ..., z_d, u)$ to $(z_1, ..., z_d, z_{d+1}, ..., z_n)$.**

DEFINITION: A meromorphic map $\varphi : X \dashrightarrow Y$ is called **bimeromorphic** if there exists a meromorphic map $\psi : Y \dashrightarrow X$ such that $\varphi \circ \psi$ and $\psi \circ \varphi$ are identities in each point where these compositions are defined.

Exceptional set of a bimeromorphic map

Let $\varphi : X \dashrightarrow Y$ be a bimeromorphic map of irreducible subvarieties of $\mathbb{C}^n, \mathbb{C}^m$, and $\psi : Y \dashrightarrow X$ the inverse map. The map φ is by definition holomorphic outside of an analytic subset $Z_X \subset X$, which is called **exceptional set of X**, and φ is holomorphic outside of an analytic subset $Z_Y \subset U$. Taking pullbacks of coordinate functions $x_1, ..., x_n, y_1, ..., y_m$ we may express φ^* as a collection of meromorphic functions $\varphi^*(y_i) = \Phi_i(x_1, ..., x_n)$ and $\psi^*(x_i) = \Psi_i(y_1, ..., y_m)$; restrictions of these functions to X, Y are holomorphic on $X \backslash Z_X$ and $Y \backslash Z_Y$. Since $\varphi \circ \psi$ and $\psi \circ \varphi$ are identity maps, we have

$$\Psi_i(\Phi_1(x_1, ..., x_n), ...\Phi_m(x_1, ..., x_n)) = x_i,$$

$$\Phi_i(\Psi_1(y_1, ..., y_n), ...\Psi_n(y_1, ..., y_n)) = y_i$$

in all points where $\Phi_i(x_1, ..., x_n)$, $\Psi_1(y_1, ..., y_n)$ are defined, that is, outside of the pole sets of these functions. These sets are complex analytic by definition. Therefore, the maps $(\Psi_1, ..., \Psi_n)$ and $(\Phi_1, ..., \Phi_m)$ induce an isomorphism between Zariski open sets of X and Y. The complements to these sets are called **the exceptional sets of φ, ψ.**

Exceptional set of a bimeromorphic map (2)

Summing it up:

CLAIM: A bimeromorphic map $\varphi : X \dashrightarrow Y$ **induces an isomorphism between a Zariski open subset of X and a Zariski open subset of Y.** When X, Y are irreducible, **a bimeromorphic map is the same as an isomorphism between the fraction fields $k(\mathcal{O}_X)$ and $k(\mathcal{O}_Y)$.**. ■

REMARK: Unlike it happens in algebraic geometry, **not every isomorphism between Zariski open sets of X and Y can be extended to a bimeromorphic map $\varphi : X \dashrightarrow Y$.** Indeed, **there are uncountably many pairwise non-bimeromorphic compact complex manifolds X_α with $\mathbb{C}^2 \subset X_\alpha$ Zariski open.**

REMARK: The difference between algebraic and complex geometry can be explained as follows. **In algebraic category, for any Zariski open set $U \subset Z$, $k(\mathcal{O}_U) = k(\mathcal{O}_Z)$, and in the complex category, $k(\mathcal{O}_U)$ is usually an extension of infinite transcendence degree over $k(\mathcal{O}_Z)$.**

Regular coordinate systems (reminder)

Consider the ring of germs of functions on $\mathbb{C}^n$ depending only on the first d coordinates. We identify this ring with $\mathcal{O}_d \subset \mathcal{O}_n$.

THEOREM: Let J be an ideal in $\mathcal{O}_n$. There exists a coordinate system $z_1, ..., z_d, z_{d+1}, ..., z_n$ in a neighbourhood of 0 such that

1. $J_d = 0$, **where** $J_d := \mathcal{O}_d \cap J$.
2. The ideal J **is generated by a collection of Weierstrass polynomials** $P_i(z_i) \in \mathcal{O}_{i-1}[z_i]$, $i = d+1, ..., n$.

DEFINITION: In this situation, $z_1, ..., z_n$ is called **a regular coordinate system for** J.

THEOREM: (Primitive element theorem)
Let J be a prime ideal in $\mathcal{O}_n$, and $z_1, ..., z_d, ..., z_n$ a regular coordinate system for J. Assume that $u := \sum_{i=d+1}^n \lambda_i z_i$ generates the fraction field $k(\mathcal{O}_n/J)$ over $k(\mathcal{O}_d)$. Define a map $\mathfrak{u}: \mathbb{C}^n \longrightarrow \mathbb{C}^{d+1}$, $\mathfrak{u}(z_1, ..., z_n) = (z_1, ..., z_d, u)$.

(1) **The map $\mathfrak{u}$ defines a holomorphic map $\varphi: Z \longrightarrow Z_u$ from the germ Z of common zeroes of J to a germ of a hypersurface $Z_u \subset \mathbb{C}^{d+1}$.**

(2) **The projection $\Pi_d: Z_u \longrightarrow \mathbb{C}^d$ to the first d coordinates is finite.**

(3) The map $Z \xrightarrow{\mathfrak{u}} Z_u$ **induces an isomorphism $k(\mathcal{O}_{d+1}/(P_u)) \xrightarrow{\sim} k(\mathcal{O}_n/J)$ on the fraction fields.**

Discriminants and coverings (reminder)

DEFINITION: Let $P(t) = \prod_i (t-\alpha_i) = t^n + a_{n-1}t^{n-1} + ... + a_0$ be a polynomial. **The discriminant** D_P of P is the product $\prod_{i<j}(\alpha_i - \alpha_j)^2$ expressed as a polynomial on the coefficients of $P(t)$: $\prod_{i<j}(\alpha_i - \alpha_j)^2 = D_P(a_{n-1}, a_{n-2}, ..., a_0)$.

REMARK: Let $P(z_n) \in \mathcal{O}_{n-1}[z_n]$ be a Weierstrass polynomial, and D_P its discriminant. Since D_P is polynomially expressed through the coefficients of P, and coefficients belong to $\mathcal{O}_{n-1}$, **we can (and will) consider the discriminant as a function on $\mathbb{C}^{n-1}$.**

Theorem 1: Let $P(z_n) \in \mathcal{O}_{n-1}[z_n]$ be a Weierstrass polynomial, $P(0, 0, ..., 0, z_n) = z^k$, $X \subset \mathbb{C}^n$ its zero set and D_P its discriminant, considered an element in $\mathcal{O}_{n-1}$. We assume that $D_P \neq 0$. Denote by Z_D the zero set of D_P, and Z_X its preimage under the projection $\Pi: X \longrightarrow \mathbb{C}^{n-1}$ to the first $n-1$ coordinates. **Then $X \backslash Z_x$ is smooth and the projection $X \backslash Z_X \xrightarrow{\Pi} \mathbb{C}^{n-1} \backslash Z_D$ is a non-ramified k-sheeted covering.**

DEFINITION: The sets Z_D and Z_X are called **the discriminant sets of Π**. The above theorem can be stated as **"Π is a non-ramified k-sheeted covering outside of its discriminant set."**

Taking average of a function over a ramified cover

The following lemma is proven by the same argument as used to prove the Weierstrass preparation theorem.

Let $P(z_n) \in \mathcal{O}_{n-1}[z_n]$ be a Weierstrass polynomial, $X \subset \mathbb{C}^n$ the germ of its zero set, and $\Pi: X \longrightarrow \mathbb{C}^{n-1}$ the projection to the first $n-1$ coordinates. Assume that $P(0,...,0,z_n) = z_n^k$ As usual, we take polydisc $\Delta(n-1,1) := B_r(z_1,...z_{n-1}) \times \Delta_{r'}(z_n)$ such that $P(z,z_n) \neq 0$ when $|z_n| = r'$; **by Rouché theorem, the set $\Pi^{-1}(z) \cap X \cap \Delta(n-1,1)$ has precisely k points, if counted with multiplicities.**

LEMMA: Consider the projection $\Pi: X \longrightarrow \mathbb{C}^{n-1}$ as above, let $z \in B_r(z_1,...z_{n-1})$, and $\{t_1,...,t_k\}$ the set $\Pi^{-1}(z)$, taken with multiplicities. Consider a function $f \in \mathcal{O}_n$, and let $\sigma(f)$ be a germ of a function on $\mathbb{C}^{n-1}$ which takes z to $\sum_{i=1}^k f(t_i)$. **Then $\sigma(f)$ is holomorphic.**

Proof: Let $\Delta_z := \Pi^{-1}(z) \cap \Delta(n-1,1)$ be the disc of radius r', obtained as the preimage of z. Denote by P_z the restriction of P to Δ_z. **Then $\sigma(f)(z) = \frac{1}{2\pi\sqrt{-1}} \int_{\partial\Delta_z} f(\zeta)\frac{P_z'(\zeta)}{P_z(\zeta)} d\zeta$.** Indeed, $\frac{P_z'(\zeta)}{P_z(\zeta)} = \frac{d}{d\zeta}\log(P_z(\zeta))$ is a function which has simple poles at t_i of the same multiplicity as t_i, hence the residues of $f(\zeta)\frac{P_z'(\zeta)}{P_z(\zeta)}$ at t_i are $2\pi\sqrt{-1}\, f(t_i)k_i$, where k_i is the multiplicity of t_i. ■

Taking average of a function over a ramified cover (2)

LEMMA: Consider the projection $\Pi: X \longrightarrow \mathbb{C}^{n-1}$ as above, let $z \in B_r(z_1, ... z_{n-1})$, and $\{t_1, ..., t_k\}$ the set $\Pi^{-1}(z)$, taken with multiplicities. Consider a function $f \in \mathcal{O}_n$, and let $\sigma(f)$ be a germ of a function on $\mathbb{C}^{n-1}$ which takes z to $\sum_{i=1}^k f(t_i)$. **Then $\sigma(f)$ is holomorphic.**

Proof: Let $\Delta_z := \Pi^{-1}(z) \cap \Delta(n-1, 1)$ be the disc of radius r', obtained as the preimage of z. Denote by P_z the restriction of P to Δ_z. **Then** $\sigma(f)(z) = \frac{1}{2\pi\sqrt{-1}} \int_{\partial\Delta_z} f(\zeta) \frac{P_z'(\zeta)}{P_z(\zeta)} d\zeta$. Indeed, $\frac{P_z'(\zeta)}{P_z(\zeta)} = \frac{d}{d\zeta} \log(P_z(\zeta))$ is a function which has simple poles at t_i of the same multiplicity as t_i, hence the residues of $f(\zeta)\frac{P_z'(\zeta)}{P_z(\zeta)}$ at t_i are $2\pi\sqrt{-1}\, f(t_i) k_i$, where k_i is the multiplicity of t_i. ■

Corollary 1: In these assumptions, let E_r be an elementary symmetric polynomial on $f(t_1), ..., f(t_k)$. **Then E_r is holomorphic as a function of z.**

Proof: Elementary symmetric polynomials on $f(t_1), ..., f(t_k)$ are expressed polynomially through the Newton polynomials $\sigma(f^l)$, where $l \in \mathbb{Z}^{\geqslant 0}$. Since $\sigma(f^l)$ is holomorphic as a function of z, the same is true about E_r. ■

Projection of a complex analytic set

REMARK: The following proposition is a weaker form of Remmert's proper image theorem: **a proper image of a complex analytic set is complex analytic.**

PROPOSITION: Let $P(z_n) \in \mathcal{O}_{n-1}[z_n]$ be a Weierstrass polynomial, $X \subset \mathbb{C}^n$ the germ of its zero set, and $\Pi : X \longrightarrow \mathbb{C}^{n-1}$ the projection to the first $n-1$ coordinates. **Then for any complex analytic set $Y \subset Z$, the projection $\Pi(Y)$ is also complex analytic.**

Proof. Step 1: Let $J \subset \mathcal{O}_n$ be the ideal of Y, and $J_{n-1} := \mathcal{O}_{n-1} \cap J$. **We are going to prove that $\Pi(Y)$ is the zero set $Z_{J_{n-1}}$ of J_{n-1}.** Clearly, any $f \in J_{n-1}$ vanishes on $\Pi(Y)$, hence $Z_{J_{n-1}} \supset \Pi(Y)$.

Step 2: Conversely, let $z \in X$ be a point such that $\Pi(z) \notin \Pi(Y)$. **We need to show that $\Pi(z) \notin Z_{J_{n-1}}$, that is, find a function $f \in J_{n-1}$ which does not vanish in z.**

Projection of a complex analytic set (2)

Step 3: Let $z = (z_1, ..., z_{n-1})$, and let $\{t_1, ..., t_k\}$ be the set $\Pi^{-1}(z)$, taken with multiplicities. Consider any $f \in \mathcal{O}_n$. Since $\xi(f) := \prod_{i=1}^k f(t_i)$ is an elementary symmetric symmetric polynomial on $f(t_1), ..., f(t_n)$, this expression is holomorphic as a function of z (Corollary 1).

For any $f \in J$, the function $\xi(f)$ vanishes in all points of $\Pi(Y)$, because it is a product of $f(t_i)$. Therefore, $f \in J_{n-1}$.

Step 4: Let $z \in X$ be a point such that $\Pi(z) \notin \Pi(Y)$. Take a function $f \notin J$ such that f does not vanish anywhere on $\Pi^{-1}(\Pi(z))$; such a function exists because X is the set of common zeros of J. Then $\xi(f) \neq 0$ in $\Pi(z)$, but $\xi(f) \in J_{n-1}$. **We have constructed a function which belongs to J_{n-1} and does not vanish in x.** ■

REMARK: The same argument also proves that **an image of a divisor is a divisor:** indeed, the image of a divisor Z_f is the zero set of the function $E_r(f)$. Moreover, **this argument works for any ramified covering map** $\varphi : X \longrightarrow Y$, obtained when X is the set of zeroes of a Weierstrass polynomial, without assuming that Y is smooth.

A complex variety always has smooth points

Let $Z \subset \mathbb{C}^n$ be a germ of a complex analytic variety, and $\varphi : Z \longrightarrow Z_u \subset \mathbb{C}^{d+1}$ the map constructed in the primitive element theorem using the regular coordinates. Since φ is bimeromorphic, it is invertible outside of the complex subvariety $E \subset Z_u$, called **the exceptional set of φ^{-1}**.

Theorem 2: In these assumptions, let $D \subset Z_u$ be the discriminant set of the projection $\Pi : Z_u \longrightarrow \mathbb{C}^d$, $E_0 \subset \mathbb{C}^d$ be $\Pi(E)$, $E_1 \subset Z$ the set $\Pi^{-1}(E_0)$, and $D_1 := \varphi^{-1}(D)$. **Then $Z \backslash (E_1 \cup D_1)$ is a smooth subset, and its projection to $\mathbb{C}^d$ a k-sheeted non-ramified covering.**

Proof: The set $Z_u \backslash (E_0 \cup D)$ is smooth by Theorem 1. **Its projection to its image in $\mathbb{C}^d$ is a k-sheeted non-ramified covering,** again by Theorem 1.

The projection $Z \backslash E_1 \xrightarrow{\varphi} Z_u \backslash E_0$ is an isomorphism, because φ is an isomorphism outside of its exceptional set. ■

The set of singularities of a complex analytic variety

PROPOSITION: In assumptions of Theorem 2, **the set $E_1 \cup D_1$ is closed and nowhere dense in Z.**

Proof. Step 1: Closedness is clear. Resricting $\Pi_d: Z \longrightarrow \mathbb{C}^d$ to a compact neighbourhood of a point, we obtain that all closed sets are compact, hence the image of a closed set is closed. Then for any open set of form $U = \Pi_d^{-1}(\Pi_d(U))$, the image $\Pi(U)$ is open; indeed, the image of its complement C is closed, and $\Pi(U)$ is the complement to $\Pi(C)$. This implies, in particular, that **a closed set $K \subset Z_u$, $K = \Pi^{-1}(\Pi(K))$ is nowhere dense if and only if its image in $\mathbb{C}^d$ is nowhere dense.**

Step 2: To show that $E_1 \cup D_1$ are nowhere dense, we notice that both sets satisfy $K = \Pi_d^{-1}(\Pi_d(K))$ and their images in $\mathbb{C}^d$ are divisors. Since the divisors in $\mathbb{C}^d$ are nowhere dense **(prove this as an exercise)**, **Step 1 implies that the divisors E_1 and D_1 are nowhere dense.** ■

The set of singularities of a complex analytic variety

DEFINITION: Dimension of an irreducible complex variety is dimension of the set of its smooth points.

REMARK: This definition is ambiguous because **we did not prove yer that this set is connected** (it is, and we will prove it). Nevertheless, **for an irreducible germ, the dimension is well defined (prove this as an exercise)**.

THEOREM: Let $Z \subset \mathbb{C}^n$ be a complex subvariety, and $Z_{sing} \subset Z$ the set of all its singular points. **Then $Z_{sing} \subset Z$ is a complex-analytic subset, and its complement is dense and open in Z.**

Proof. Step 1: Since the result is local, it suffices to prove it assuming that Z is a germ of a subvariety. Choose regular coordinates, and let $E_1 \cup D_1$ be the set defined in Theorem 2. **Outside of $E_1 \cup D_1$, the variety Z is smooth, and $E_1 \cup D_1$ is nowhere dense.**

Step 2: Let $f_1, ..., f_n$ be the generators of the ideal of germs of holomorphic functions vanishing in Z. Then **Z_{sing} is the set of all $z \in Z$ where the rank of the matrix $\langle df_1, ..., df_n \rangle$ is strictly less than $\operatorname{codim} Z$**, hence it is complex analytic. ■