Complex analytic spaces

lecture 13: Finite morphisms

Misha Verbitsky

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Finite homomorphism of the rings

DEFINITION: Let $A \longrightarrow B$ be a homomorphism of rings. It is called **finite** if B is finitely generated as A-module.

EXAMPLE: Let $J \subset A$ be an ideal. Then **the quotient map $A \longrightarrow A/J$ is finite.**
EXAMPLE: $\mathbb{Z}[\sqrt{-1}\,]$ **is finite over $\mathbb{Z}$.**
EXAMPLE: (Finiteness theorem)
Let $z_1, ..., z_n$ be a regular coordinate system for an ideal $J \subset \mathcal{O}_n$, and $\mathcal{O}_d$ holomorphic functions, depending only on $z_1, ..., z_d$. Then **the ring $\mathcal{O}_n/J$ is finitely generated as a $\mathcal{O}_d$-module.**
Proof: Lecture 8. ■

In other words, **the ring morphism $\mathcal{O}_d \longrightarrow \mathcal{O}_n/J$ is finite.**

DEFINITION: Let $A \longrightarrow B$ be an injective, finite morphism of Noetherian rings, where B has no zero divisors, and $x \in B$. The chain $1, \langle 1, x\rangle, \langle 1, x, x^2\rangle, ...$ stabilizes by Noetherianity, which gives a polynomial relation in B: $P(x) = x^n + a_{n-1}x^{n-1} + ... + a_0 = 0$, where all $a_i \in A$. The polynomial $P(x)$ is called **the minimal polynomial for x.**

EXERCISE: Let $A \longrightarrow B$ and $B \longrightarrow C$ be finite ring homomorphisms. **Prove that the composition $A \longrightarrow C$ is also finite.**

Dominant morphisms

DEFINITION: Let Z, Z_1 be germs of complex varieties. Recall that **a morphism** $\varphi: Z \longrightarrow Z_1$ is a germ of a map which can is given in coordinates by holomorphic functions: if $Z \subset \mathbb{C}^n$ and $Z_1 \subset \mathbb{C}^m$ are germs of complex analytic subsets, **φ is expressed by an m-tuple of holomorphic functions $\varphi_1, ..., \varphi_m \in \mathcal{O}_n$ taking $z \in Z$ to $(\varphi_1(z), ..., \varphi_m(z)) \in Z_1$.**

DEFINITION: In this situation, the ring $\mathcal{O}_Z$ of germs of functions on Z is a $\mathcal{O}_{Z_1}$-module. We say that φ is **finite** if $\mathcal{O}_Z$ is finitely generated as a $\mathcal{O}_{Z_1}$-module, and **dominant** if $\varphi(Z)$ does not belong to a proper complex-analytic subvariety of Z_1.

CLAIM: A morphism $\varphi: Z \longrightarrow Z_1$ **is dominant if and only if the corresponding ring morphism $\varphi^* \mathcal{O}_{Z_1} \longrightarrow \mathcal{O}_Z$ is injective.**

Proof: If $\ker \varphi^* \ni f \neq 0$, the image of φ belongs to the zero-set $Z_{(f)}$ of an ideal (f). This zero-set is non-trivial by Rückert Nullstellensatz. Conversely, if the image of φ belongs to a proper subvariety $Z_2 \subsetneq Z_1$, and f vanishes on Z_2, then $f \in \ker \varphi^*$. ■

Finiteness theorem and discriminants

Recall the following theorem (Lecture 8)

THEOREM: (Finiteness theorem)

Let $z_1, ..., z_n$ be a regular coordinate system for an ideal $J \subset \mathcal{O}_n$, and $\mathcal{O}_d$ holomorphic functions, depending only on $z_1, ..., z_d$. Then **the ring $\mathcal{O}_n/J$ is finitely generated as a $\mathcal{O}_d$-module.**

Proof: Choose regular coordinates $z_1, ..., z_d, z_{d+1}, ..., z_n$ on Z. Then the ring $\mathcal{O}_Z$ is generated over $\mathcal{O}_d$ by a collection of coordinate functions which all satisfy equations $P_i(z_i) = 0$ where P_i is a monic polynomial, $P_i \in \mathcal{O}_i[z_i]$, $i = d+1, d+2, ..., n$. ■

THEOREM: In these assumptions, let $J_r := \mathcal{O}_r \cap J$, and let $Z_r \subset \mathbb{C}^r$ be the set of common zeros of J_r. Let D_r be the discriminant of the polynomial $P_r[z_r]$ restricted to Z_{r-1}. Then

(i) the coordinate projection $\pi_r : Z_r \longrightarrow Z_{r-1}$ to the first $r-1$ coordinates is a surjective map

(ii) any point of Z_{r-1} has a finite number of preimages,

(iii) the images of the coordinate projections $\Pi_r(D_r) \subset \mathbb{C}^d$ to the first d variables are divisors

(iv) and outside of $\pi(D_r) \subset Z_{r-1}$ the number of preimages $\#(\pi_{r-1}^{-1}(z))$ is deg P_r.

Finiteness theorem and discriminants (2)

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(i) the coordinate projection $\pi_r : Z_r \longrightarrow Z_{r-1}$ to the first $r-1$ coordinates is a surjective map
(ii) any point of Z_{r-1} has a finite number of preimages,
(iii) the images of the coordinate projections $\Pi_r(D_r) \subset \mathbb{C}^d$ to the first d variables are divisors
(iv) and outside of $\pi(D_r) \subset Z_{r-1}$ the number of preimages $\#(\pi_{r-1}^{-1}(z))$ is $\deg P_r$**.**

Proof. Step 1: (i) and (ii) follow from the same "Rouché theorem" argument we used many times already: since $\pi_r : Z_r \longrightarrow Z_{r-1}$ is the set of solutions of the equation $P_r(z, z_r) = 0$, around zero it has $\deg P_r$ preimages, counted with multiplicities.

(iii) is clear, because the discriminant is a divisor, and an image of a divisor under a finite regular projection is also a divisor (Lecture 12).

To prove (iv), we notice that for any given z near the origin, the roots of the equation $P_r(z, z_r) = 0$ are multiple if and only if $D_r(z) = 0$. ■

Finite morphisms

COROLLARY: Let Z be a germ of a complex analytic variety, and $z_1, ..., z_d, z_{d+1}, ..., z_n$ the regular coordinates. By definition, the ideal of Z is generated by Weierstrass polynomials $P_i \in \mathcal{O}_i[z_i]$, $i = d+1, d+2, ..., n$. **Then, outside of the union of the images of the discriminant divisors, the projection $Z \longrightarrow \mathbb{C}^d$ to the first d coordinates is a covering of order $\prod_{i=d+1}^n \deg P_i$.** Moreover, **these divisors are nowhere dense.**

Proof: Use the induction in n and apply the previous theorem, part (iv). ■

THEOREM: Let $\varphi: X \longrightarrow Y$ be a finite, dominant morphism of the germs of complex varieties. Then **φ is a finite covering outside of a proper complex analytic subset of Y.**

Proof. Step 1: Choose the regular coordinates $z_1, ..., z_d, ..., z_n$ for Y. Then $\mathcal{O}_d$ is a subring of $\mathcal{O}_Y$. Since φ is dominant, $\varphi^*(\mathcal{O}_d)$ is a subring of $\mathcal{O}_X$. The ring $\mathcal{O}_Y$ is generated over $\mathcal{O}_d$ by a collection of Weierstrass polynomials $P_i \in \mathcal{O}_i[z_i]$, $i = d+1, d+2, ..., n$. The ring $\mathcal{O}_X$ is generated over $\mathcal{O}_Y$ by a collection of elements u_i which satisfy polynomial equations $Q_i[u_i] = 0$, $Q_i[t] \in \mathcal{O}_X(t)$ with leading term 1. Taking these u_i as extra coordinates, **we obtain a compatible system of regular coordinates on X and Y,** in such a way that **Y is obtained as a projection of Y to the first n coordinates.**

Step 2: The projection of X to the first n coordinates **is a covering outside of the union of the discriminant varieties,** as shown above. ■