Complex analytic spaces

lecture 14: Dimension

Misha Verbitsky

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Finite and dominant morphisms (reminder)

DEFINITION: Let $A \longrightarrow B$ be a homomorphism of rings. It is called **finite** if B is finitely generated as A-module.

EXAMPLE: (Finiteness theorem)
Let $z_1, ..., z_n$ be a regular coordinate system for an ideal $J \subset \mathcal{O}_n$, and $\mathcal{O}_d$ holomorphic functions, depending only on $z_1, ..., z_d$. Then **the ring $\mathcal{O}_n/J$ is finitely generated as a $\mathcal{O}_d$-module.** In other words, **the ring morphism $\mathcal{O}_d \longrightarrow \mathcal{O}_n/J$ is finite.**

DEFINITION: Let Z, Z_1 be germs of complex varieties. Recall that **a morphism** $\varphi : Z \longrightarrow Z_1$ is a germ of a map which can is given in coordinates by holomorphic functions: if $Z \subset \mathbb{C}^n$ and $Z_1 \subset \mathbb{C}^m$ are germs of complex analytic subsets, **φ is expressed by an m-tuple of holomorphic functions $\varphi_1, ..., \varphi_m \in \mathcal{O}_n$ taking $z \in Z$ to $(\varphi_1(z), ..., \varphi_m(z)) \in Z_1$.**

DEFINITION: In this situation, the ring $\mathcal{O}_Z$ of germs of functions on Z is a $\mathcal{O}_{Z_1}$-module. We say that φ is **finite** if $\mathcal{O}_Z$ is finitely generated as a $\mathcal{O}_{Z_1}$-module, and **dominant** if $\varphi(Z)$ does not belong to a proper complex-analytic subvariety of Z_1.

CLAIM: A morphism $\varphi : Z \longrightarrow Z_1$ **is dominant if and only if the corresponding ring morphism $\varphi^* \mathcal{O}_{Z_1} \longrightarrow \mathcal{O}_Z$ is injective.**

Finiteness theorem and discriminants (reminder)

THEOREM: (Finiteness theorem)
Let $z_1, ..., z_n$ be a regular coordinate system for an ideal $J \subset \mathcal{O}_n$, and $\mathcal{O}_d$ holomorphic functions, depending only on $z_1, ..., z_d$. Then **the ring $\mathcal{O}_n/J$ is finitely generated as a $\mathcal{O}_d$-module.**

Proof: Choose regular coordinates $z_1, ..., z_d, z_{d+1}, ..., z_n$ on Z. Then the ring $\mathcal{O}_Z$ is generated over $\mathcal{O}_d$ by a collection of coordinate functions which all satisfy a polynomial equation $P(z_i) = 0$ where P_i is a monic polynomial, $P_i \in \mathcal{O}_i[z_i]$, $i = d+1, d+2, ..., n$. ■

THEOREM: In these assumptions, let $J_r := \mathcal{O}_r \cap J$, and let $Z_r \subset \mathbb{C}^r$ be the set of common zeros of J_r. Let D_r be the discriminant of the polynomial $P_r[z_r]$ restricted to Z_{r-1}. Then
(i) the coordinate projection $\pi_r : Z_r \longrightarrow Z_{r-1}$ to the first $r-1$ coordinates is a surjective map
(ii) any point of Z_{r-1} has a finite number of preimages,
(iii) the images of the coordinate projections $\Pi_r(D_r) \subset \mathbb{C}^d$ to the first d variables are divisors
(iv) and outside of $\pi(D_r) \subset Z_{r-1}$ the number of preimages $\#(\pi_{r-1}^{-1}(z))$ is deg P_r.

Dimension

DEFINITION: Recall that the **dimension** of a germ of an irreducible complex variety is the dimension of the set of its smooth points (assuming it is constant, which is proven below).

DEFINITION: A variety is called **equidimensional**, if all its irreducible components have the same dimension.

PROPOSITION: Let $\varphi: X \longrightarrow X_1$ be a finite, dominant map of complex varieties. **Then** $\dim X = \dim X_1$**.**

Proof: We have shown that φ is a covering outside of the union of discriminant divisors. Therefore, $\dim X = \dim X_1$. ■

REMARK: In the definition of the dimension, there was an ambiguity related to possible existence of open, smooth subsets of different dimension in the same germ of an irreducible variety. **This ambiguity is resolved**, because **a finite, dominant map preserves the dimension,** and **any germ of a variety admits a finite map to** $\mathbb{C}^d$**,** which is **unambiguously** d**-dimensional.**

Dimension of a divisor

LEMMA: Let $\varphi : X \longrightarrow Y$ be a finite map, and $Z \subset X$ is a subvariety. Then $f : Z \longrightarrow \varphi(Z)$ **is finite and dominant.**

Proof: This map is dominant by definition; it is finite because the quotient map $\mathcal{O}_X \longrightarrow \mathcal{O}_Z$ is finite, hence the composition $\psi : \mathcal{O}_Y \longrightarrow \mathcal{O}_Z$ is also finite. Finally, $\mathcal{O}_{\varphi(Z)}$ is, also by definition, equal to $\frac{\mathcal{O}_Y}{\ker \psi}$. ■

THEOREM: Let Z be an irreducible germ of a variety, and $Z_f \subset Z$ the germ of a zero divisor of a non-trivial function. **Then** $\dim Z_f = \dim Z - 1$**.**

Proof. Step 1: Suppose first that $Z = \mathbb{C}^n$. Applying the Weierstrass preparation theorem, we can represent f as a Weierstrass polynomial. **Then the coordinate projection of Z_f to $\mathbb{C}^{n-1}$ is a covering outside of its discriminant,** hence $\dim Z_f = n - 1$.

Step 2: Consider the finite, dominant map $\pi : Z \longrightarrow \mathbb{C}^d$ associated with the regular coordinates. **The image of a divisor is a divisor (Lecture 12), hence** $\dim(\pi(Z_f)) = d - 1$ **by Step 1.** On the other hand, $\dim Z_f = \dim \pi(Z_f)$, because $\pi : Z_f \longrightarrow \pi(Z_f)$ is finite and dominant. Therefore, $\dim Z_f = d - 1 = \dim Z - 1$. ■

Dimension of a divisor: useful observations

CLAIM: An irreducible complex variety X **is never equal to a countable union of its divisors**. Also, **divisors are nowhere dense.**

Proof: Indeed, a divisor has measure 0 in the set of smooth points of X. ■

COROLLARY: A meromorphic function **is holomorphic outside of its pole**, which is a closed, nowhere dense subvariety. ■

COROLLARY: If $X \subset Y$ are complex analytic, then $\dim X \leqslant Y$ (prove it).

COROLLARY: $\dim X \geqslant \dim X_{sing}$.

Krull dimension

DEFINITION: Let X be an irreducible variety, and $X_n \subsetneq X_{n-1} \subsetneq ... \subsetneq X_1 \subsetneq X$ a chain of irreducible subvarieties such that each X_i is a divisor in X_{i-1}. Then the number n is called **the Krull dimension** of X.

THEOREM: The Krull dimension of an irreducible variety is equal to its dimension.

Proof. Step 1: Using the regular coordinates, choose a finite, dominant map $\pi: X \longrightarrow \mathbb{C}^m$ locally in a neighbourhood of a given point. Since a finite and dominant map takes divisors to divisors and preserves the dimension, the map π preserves the Krull dimension and the dimension. Therefore, **it suffices to prove the theorem when $X = \mathbb{C}^n$.**

Step 2: Using induction, we may assume we proved the theorem when $\dim X \leqslant n-1$. Choose a divisor Z on $\mathbb{C}^n$. Then $\dim Z = n-1$ because it has smooth points, and its Krull dimension is $n-1$ by induction assumption. **This implies that the Krull dimension of $\mathbb{C}^n$ is n.** ■