Complex analytic spaces

lecture 15: The maximum principle and the maps with finite fibers

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Finite homomorphism of the rings (reminder)

DEFINITION: Let $A \longrightarrow B$ be a homomorphism of rings. It is called **finite** if B is finitely generated as A-module.

EXAMPLE: Let $J \subset A$ be an ideal. Then **the quotient map $A \longrightarrow A/J$ is finite.**
EXAMPLE: $\mathbb{Z}[\sqrt{-1}\,]$ **is finite over $\mathbb{Z}$.**
EXAMPLE: **(Finiteness theorem)**
Let $z_1, ..., z_n$ be a regular coordinate system for an ideal $J \subset \mathcal{O}_n$, and $\mathcal{O}_d$ holomorphic functions, depending only on $z_1, ..., z_d$. Then **the ring $\mathcal{O}_n/J$ is finitely generated as a $\mathcal{O}_d$-module.**
Proof: Lecture 8. ■

In other words, **the ring morphism $\mathcal{O}_d \longrightarrow \mathcal{O}_n/J$ is finite.**

DEFINITION: Let $A \longrightarrow B$ be an injective, finite morphism of Noetherian rings, where B has no zero divisors, and $x \in B$. The chain $1, \langle 1, x\rangle, \langle 1, x, x^2\rangle, ...$ stabilizes by Noetherianity, which gives a polynomial relation in B: $P(x) = x^n + a_{n-1}x^{n-1} + ... + a_0 = 0$, where all $a_i \in A$. The polynomial $P(x)$ is called **the minimal polynomial for x.**

EXERCISE: Let $A \longrightarrow B$ and $B \longrightarrow C$ be finite ring homomorphisms. **Prove that the composition $A \longrightarrow C$ is also finite.**

Averaging function on the preimages under a finite map (reminder)

Using the same arguments as in Lecture 12, we prove the following proposition.

PROPOSITION: Let $X \longrightarrow Y$ be a finite map of germs of varieties, obtained as a ramified covering of order k, and f a germ of a function on X. Denote by $\sigma(f)$ the function on Y which takes $y \in Y$ to the sum $\sum_{\{} t_i\} f(t_i)$, where $\{t_i\}$ the k-tuple of preimages of y taken with multiplicities. **Then $\sigma(y)$ is holomorphic.**

Proof. Step 1: As shown in Lecture 13, any finite map is obtained as a composition of coordinate projections $\pi_n: A \to B$, where $A \subset \mathbb{C}^n$, $B \subset \mathbb{C}^{n-1}$, and the ideal J_A of A is obtained from J_B by adding a Weierstrass polynomial $P_n \in \mathcal{O}_{n-1}[z_n]$. **Then $\varphi^{-1}(y)$ is identified with the set of roots of P_n on the 1-dimensional disk $\Delta_y := \{u \in \pi_n^{-1}(y) \ \ | \ \ |y| \leqslant r'\}$.**

Step 2: The function $\sigma(f)$ is written as a Cauchy integral $\sigma(f)(y) = \frac{1}{2\pi\sqrt{-1}} \int_{\partial\Delta_y} f(\zeta)\frac{P_n'(\zeta)}{P_n(\zeta)} d\zeta$ (Lecture 12). ■

Exercise 1: Let $\alpha_1, ..., \alpha_k$ be a collection of complex numbers, and α their average, $\alpha = \frac{1}{k}\sum_{i=1}^k \alpha_i$. Suppose that $|\alpha| \geqslant \max|\alpha_i|$. **Prove that $\alpha = \alpha_i$, for all i.**

The maximum principle

LEMMA: Let $U \subset \mathbb{C}$ be a connected open set, and $f: U \longrightarrow \mathbb{C}$ a non-constant function. **Then f is open**, that is, maps an open set to an open set.

Proof. Step 1: It would suffice to prove that for any $x \in U$, $f(x) = y$, there is a neighbourhood $V \ni y$ such that $V \subset f(U)$.

Step 2: Choose a small disk Δ centered in x such that $f(z) - y$ does not vanish for all $z \in \partial\Delta$. Let $c := \inf_{z \in \partial\Delta} |f(z) - y|$. Then for all $t \in \mathbb{C}$ such that $|t| < c$, **the number of preimages of zero of a function $f(z) - y - t$ is the same as of $f(z) - y$, by Rouché theorem.** This gives $y + t \in f(U)$. ■

COROLLARY: (Maximum principle)
Let $U \subset \mathbb{C}^n$ be a connected open subset, and $f: U \longrightarrow \mathbb{C}$ a non-constant function. **Then $|f|$ does not reach a local maximum on U.**
Proof. Step 1: It would suffice to prove that for any $x \in U$, $f(x) = y$, there is a neighbourhood $V \ni y$ such that $V \subset f(U)$.

Step 2: Since f is non-constant, there exist a point x' in a convex neighbourhood of x in U such that $f(x) \neq f(x')$. Denote by Δ a disk on a complex line connecting x to x'. Then $f|_\Delta$ is non-constant, and existence a neighbourhood $V \ni y$ such that $V \subset f(U)$ is guarantied by the previous lemma.
■

The maximum principle for complex varieties

THEOREM: (the maximum principle)
Let f be a holomorphic function on a connected complex variety X. **Assume that $|f|$ attains a local maximum somewhere on X. Then f is constant.**

Proof. Step 1: Let $x \in X$ be a point where $|f(x)|$ attains a local maximum. Construct a finite projection π of a neighbourhood of X to $\mathbb{C}^d$, such that the preimage of 0 is x with multiplicity k. By the previous exercise $|k^{-1}\sigma(f)(y)| \leqslant \max_{t_i \in \pi^{-1}(y)} |f(t_i)|$, and the inequality is strict only when all $f(t_i)$ are equal. Therefore, $|k^{-1}\sigma(f)(0)| = |f(x)|$ reaches its local maximum in 0. By maximum principle for a smooth subsets of $\mathbb{C}^d$, the function $\sigma(f)$ is constant.

Step 2: Now, for each y sufficiently close to x, we have

$$\max_{t_i \in \pi^{-1}(y)} |f(t_i)| \leqslant |f(x)| = |k^{-1} \max_{t_i \in \pi^{-1}(y)} |f(t_i)|.$$

Applying the previous exercise again, we obtain that $f(t_i) = f(x)$. ■
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COROLLARY: Let $Z \subset \mathbb{C}^n$ be a compact, connected complex submanifold. **Then Z is a point.**
Proof: Indeed, the coordinate functions on Z attain a maximum, hence they are constant. ■

Constant rank theorem

DEFINITION: Let X be a complex manifold, and $F: X \longrightarrow \mathbb{C}^n$ a holomorphic map. **Rank of F in $x \in X$** is the number $\operatorname{rk}_x F := \dim X - \dim \ker dF$.

THEOREM: (Constant rank theorem)
Let $F: X \longrightarrow Y$ be a holomorphic map of complex varieties, where X is smooth. Assume that F has constant rank. **Then $F^{-1}(y)$ is a smooth submanifold of X for all $y \in F(X)$.** Moreover, **for any $x \in X$ there is a neighbourhood $U \ni x$ such that $F(U)$ is a smooth manifold of dimension $\operatorname{rk} F$.**

Proof. Step 1: If $\operatorname{rk} F = \dim X = n$ in $x \in X$, we compose F with a coordinate projection in order to obtain a map $F_1: X \longrightarrow \mathbb{C}^n$ with dF_1 invertible. **Then F_1 defines a local diffeomorphism of a neighbourhood $U \subset X$ of x to an open subset $W \subset \mathbb{C}^n$ by the inverse function theorem.** Therefore, $F(U)$, which is biholomorphic to W, is a smooth subvariety in Y.

Step 2: If $\operatorname{rk} F = k < \dim X = n$, we replace X by a sufficiently small neighbourhood of x, and take another map $G: X \longrightarrow \mathbb{C}^{n-k}$ such that dG is invertible on the space $\ker dF$ (which is $(n-k)$-dimensional). Then $F \times G: X \longrightarrow Y \times \mathbb{C}^{n-k}$ is a constant rank map which satisfies $\operatorname{rk} F \times G = \dim X$, and **Step 1 implies that it is biholomorphic to its image which is smooth and locally isomorphic to $\operatorname{im} F \times \mathbb{C}^{n-k}$.** Then $\operatorname{im} F$ is also smooth, and preimages of any $y \in F(Y)$ are smooth by the inverse function theorem. ∎

Maps with finite fibers

We know that finite maps have finite fibers and preserve the dimension. Now we can prove that the finiteness assumption can be dropped: **all maps with finite fibers preserve the dimension.**

LEMMA: Let $F: X \longrightarrow Y$ be a holomorphic map of complex varieties. Assume that $F^{-1}(y)$ is finite for all $y \in Y$. **Then** $\dim X \leqslant \dim Y$**.**

Proof. Step 1: Using the finiteness theorem, we construct a finite map from Y to a disk U of the same dimension. This map has finite fibers. **Replacing** Y **by** U**, we may assume that** Y **is a disk in** $\mathbb{C}^n$.

Step 2: Let $x \in X$ be a point where dF has maximal rank k. Clearly, $\operatorname{rk} dF \geqslant k$ is equivalent to a non-vanishing of a $k \times k$ minor in the matrix dF. Locally in a neighbourhood of x, this minor is non-zero, hence $\operatorname{rk} dF = k$ in a neighbourhood U of x. Since the singular set is nowhere dense, we can always assume that U is smooth. **By the constant rank theorem,** $F(X)$ **is smooth and has the same dimension as** X**, which implies** $\dim X \leqslant \dim Y$**.** ■

Dimension of an intersection with a subspace

Lemma 1: Let $Z \subset \mathbb{C}^n$ be an equidimensional subvariety, and $V \subset \mathbb{C}^n$ a k-dimensional subset. Assume that $\dim(V \cap Z) = 0$. **Then** $\dim Z \leqslant n - k$**.**

Proof: As shown in Lecture 14, $\dim Z_f = \dim Z - 1$ for any divizor $Z_f \subset Z$. Then induction gives $\dim(Z \cap V) \geqslant \dim Z - \operatorname{codim} V$. ■

Lemma 2: Let $Z \subset \mathbb{C}^n$ be an equidimensional subvariety of dimension d. Then for all $(n-d)$-dimensional affine subspaces V outside of a measure zero set of the relevant Grassmannian, **the intersection $Z \cap V$ has dimension 0.**

Proof. Step 1: Covering Z by a countable collection of open charts U_i, and proving the result for these charts, we obtain that the set of all V such that $Z \cap V$ has dimension 0 is a complement to a countable union of measure 0 sets. Therefore, it suffices to prove Lemma 1 locally for a sufficiently small open set. **We reduced Lemma 2 to a statement about a germ of a variety.**

Step 2: We assume that Z is a germ of a subvariety of $\mathbb{C}^n$ in 0. The set of linear coordinate frames is naturally identified with $GL(n, \mathbb{C})$. **Outside of a measure zero set in $GL(n, \mathbb{C})$, all linear coordinate systems $z_1, ..., z_n$ are regular.** Then the linear projection π_d to $\mathbb{C}^d$ is finite, and every point in $\mathbb{C}^d$ has finite preimage in Z. Outside of a measure 0 set, any $(n-d)$-dimensional subspace $V \subset \mathbb{C}^n$ can be realized as a preimage $\pi_d^{-1}(z)$ for an appropriate regular coordinate system, and $Z \cap V$ has dimension 0 for all such V. ■

Maps with finite fibers: semicontinuity

(*) **PROPOSITION:** Let $F:\ \mathbb{C}^n \longrightarrow \mathbb{C}^m$ be a holomorphic submersion preserving 0, and $Z \subset \mathbb{C}^n$ a germ of a subvariety in 0, satisfying $F^{-1}(0) \cap Z = 0$. **Then F is proper in a neighbourhood of 0 and has finite fibers**.

Proof: Performing a local coordinate change, we may always assume that F is a linear projection. Let $U = D \times D'$ be a polydisc neighbourhood of $0 \in \mathbb{C}^n$, where F projects $D \times D'$ to D' along D. **Choosing D' sufficiently small, we can assume that $Z \cap \partial D \times D' = \emptyset$.** Indeed, $Z \cap \partial D \times D'$ is closed, and its intersection with $F^{-1}(0)$ is empty.

Step 2: Let $t \in D'$. Since $F^{-1}(t) \cap Z$ is a closed subset not intersecting the boundary, it is compact in $D \times \{t\}$. Then the restriction $F|_{Z \cap D \times D'}:\ Z \cap D \times D \longrightarrow D'$ is proper. **The fibers of F are compact closed subvarieties of a disk; they are finite by maximum principle**. ■