Complex analytic spaces

lecture $16\frac{1}{2}$: Finite group quotients

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Group representations

DEFINITION: Representation of a group G is a homomorphism $G \longrightarrow GL(V)$. In this case, V is called **representation space**, and **a representation**.

DEFINITION: Irreducible representation is a representation having no G-invariant subspaces. **Semisimple representation** is a direct sum of irreducible ones.

Let V be a vector space over a field k. **The space of bilinear maps $V \times V \longrightarrow k$ is denoted $V^* \otimes V^*$.**

REMARK: If the group G acts on a vector space V, it **G acts on $V^* \otimes V^*$ as $g(h)(x,y) = h(g^{-1}(x), g^{-1}(y))$**, for any $g \in G$, $h \in V^* \otimes V^*$ and $x, y \in V$.

DEFINITION: A metric h (Euclidean or Hermitian) on a vector space V is called **G-invariant** if the corresponding tensor $h \in V^* \otimes V^*$ is G-invariant.

G-invariant metrics

CLAIM:
A sum of two Hermitian (Euclidean) metrics is Hermitian (Euclidean). ■

COROLLARY: Let V be a representation of a finite group (over $\mathbb{R}$ or $\mathbb{C}$). **Then V admits a G-invariant metric** (Hermitian or Euclidean).

Proof: Let h be an arbitrary metric, and $\frac{1}{|G|}\sum_{g\in G} g(h)$ its average over the G action. The previous claim implies that it is a metric. Since G acts on itself bijectively, interchanging all terms in the sum, **it is G-invariant.** ■

COROLLARY: Let $E \subset V$ be a subrepresentation in a finite group representation over $\mathbb{R}$ or $\mathbb{C}$. Then **V can be decomposed onto a direct sum of two G-representations $V = W \oplus W'$.**

Proof: Choose a G-invariant metric on V, and let $W^\perp$ be the orthogonal complement to W. Then $W^\perp$ is also G-invariant **(check this)**. This gives a decomposition $V = W \oplus W^\perp$. ■

COROLLARY: Any finite-dimensional representation of a finite group is semisimple. ■

Exact functors

DEFINITION: An **exact sequence** is a sequence of vector spaces and maps $... \longrightarrow A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow ...$ such the kernel of each map is the image of the previous one. A **short exact sequence** is exact sequence of form $0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$. **Here "exact" means that i is injective, j surjective, and the image of i is the kernel of j.**

DEFINITION: A functor $A \longrightarrow FA$ on the category of R-modules or vector spaces is called **left exact** if any exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is mapped to an exact sequence

$$0 \longrightarrow FA \longrightarrow FB \longrightarrow FC,$$

right exact if it is mapped to an exact sequence

$$FA \longrightarrow FB \longrightarrow FC \longrightarrow 0,$$

and **exact** if the sequence

$$0 \longrightarrow FA \longrightarrow FB \longrightarrow FC \longrightarrow 0$$

is exact.

Invariants and coinvariants

DEFINITION: Let G be a finite group, and V its representation. Define **the space of G-invariants** V^G as the space of all G-invariant vectors, and **the space of coinvariants** as the quotient of V by its subspace generated by vectors $v - g(v)$, where $g \in G, v \in V$.

CLAIM: Let V be an irreducible representation of G. **Then its invariants and co-ivariants are equal 0 if it is non-trivial, and equal V if it is trivial.**

COROLLARY: Let V be a semisimple representation of G. **Then $V_G = V^G$.**

EXERCISE: Prove that **the functor $V \longrightarrow V^G$ is left exact, and $V \longrightarrow V_G$ is right exact**.

COROLLARY: For any finite group G, the functor of G-invariants $V \longrightarrow V^G$ on the category of complex representations of G is exact.

REMARK: The averaging map

$$m \longrightarrow \frac{1}{|G|} \sum_{g \in G} g(m)$$

gives a projection of V to V^G, and the kernel of this map is the kernel of the natural projection $V \longrightarrow V_G$

Semisimplicity of representations of finite groups

PROPOSITION: Let $\mathcal{R}ep_k(G)$ be the category of representations of a finite group G over a field k, with $\operatorname{char}(k)$ coprime with $|G|$. **Then any short exact sequence of G-representations splits.**

Proof. Step 1: Let $0 \longrightarrow A \stackrel{i}{\longrightarrow} B \stackrel{j}{\longrightarrow} C \longrightarrow 0$ be an exact sequence of G-representations. Choose a basis $\{z_i\}$ in C, and let $\{\tilde{z}_i\}$ be preimages of z_i in B. Axiom of Choice gives a way to chose these preimages even if the set $\{z_i\}$ is infinite. Let $\varphi: C \longrightarrow B$ take z_i to $\tilde{z}_i$. Then $B = i(A) \oplus \varphi(C)$. However, **this does not imply that (*) splits, because the map φ is not necessarily G-invariant, and the space $\varphi(C)$ is not necessarily a subrepresentation.**

Step 2: We are going to modify φ such that it becomes G-invariant. Consider the action of G on $\operatorname{Hom}(C, B)$ taking $g \in G$ and $u \in \operatorname{Hom}(C, B)$ to $gug^{-1} \in \operatorname{Hom}(C, B)$; here the first "$g$" denotes the corresponding element in $GL(B)$ and the "g^{-1}" denotes the element in $GL(C)$. **Then φ is a morphism of G-representations if and only if φ is G-invariant.**

Semisimplicity of representations of finite groups (2)

PROPOSITION: Let $\mathcal{R}ep_k(G)$ be the category of representations of a finite group G over a field k, with $\operatorname{char}(k)$ coprime with $|G|$. **Then any short exact sequence of G-representations splits.**

Proof. Step 1: Let $0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$ be an exact sequence of G-representations. Consider j as a surjection of vector spaces and **find a section $\varphi: C \longrightarrow B$ (not necessarily G-invariant) using a basis in C.**

Step 2: To split this exact sequence of representations, φ **should be chosen G-invariant.**

Step 3: Since $\operatorname{char} k$ is coprime with $|G|$, the number $|G|$ is invertible in k. Let $\varphi_0 := \frac{1}{|G|}\sum_{g\in G} g(\varphi)$. **This is a sum of all elements in a G-orbit,** hence it is G-invariant. For any $v \in C$, one has

$$i(\varphi_0(v)) = \frac{1}{|G|}\sum_{g\in G} j(g(\varphi))(g^{-1}v) = \frac{1}{|G|}\sum_{g\in G} g(j\varphi((g^{-1}v))) = \frac{1}{|G|}\sum_{g\in G} g(g^{-1}(v)) = v,$$

because j commutes with φ. This implies that **φ_0 is a G-invariant section of j.** ■

Finite group action admits a linearization

THEOREM: Let G be a finite group acting on $U \subset \mathbb{C}^n$ by holomorphic automorphisms preserving 0. **Then there exist a coordinate system in which G acts linearly.**

Proof. Step 1: Consider the map $\delta: \mathcal{O}_U \longrightarrow T_0^*U$ taking f to $df|_0$. **It would suffice to show that there exists a finite-dimensional G-invariant subspace $W \subset \mathcal{O}_U$ such that $\delta: W \longrightarrow T_0^*U$ is an isomorphism.** Indeed, the inverse function theorem would imply that the map $\varphi_W: W \longrightarrow \mathcal{O}_U$, taking the basis vectors $\xi_1, ..., \xi_n \in W$ to their images in $\mathcal{O}_U$ defines a coordinate system, and G acts on the image of this map linearly.

Step 2: The exact sequence of G-representations $0 \longrightarrow \ker\delta \longrightarrow \mathcal{O}_U \xrightarrow{\delta} T_0^*U \longrightarrow 0$ splits, as shown above. **Then there exists a G-equivariant map $T_0^*U \longrightarrow \mathcal{O}_U$ inverting δ.** ■

REMARK: We are going to show that the quotient U/G is well defined as a complex variety. By the previous theorem, **it would suffice to construct the quotient in the category of affine algebraic varieties.**

Noether theorem (scheme of the proof)

THEOREM: Let R be a finitely generated ring over $\mathbb{C}$, and G a finite group acting on R by automorphisms. Then **the ring R^G of G-invariants is finitely generated**.

Scheme of the proof:

1. Noetheriannes of R is used to prove that R^G is Noetherian.

2. Prove that R^G is finite generated for $R = \mathbb{C}[z_1, ..., z_n]$, where R acts on polynomials of degree 1 by linear automorphisms.

3. Deduce the general case from (2) and exactness of $V \longrightarrow V^G$

Ideals in R and R^G

LEMMA: Let R be a ring, G a finite group acting on R, R^G the ring of G-invariants, and $I \subset R^G$ an ideal. Then **the ideal RI satisfies $\operatorname{Av}_G(RI) = \operatorname{Av}_G(R)I = R^G I = I$,** where $\operatorname{Av}_G : R \longrightarrow R^G$ denotes the averaging map.

Proof: $\operatorname{Av}_G(xy) = \operatorname{Av}_G(x)y$ if y is G-invariant. ■

COROLLARY: Let $I_1 \subsetneq I$ be ideals in R^G. **Then $RI_1 \subsetneq RI$.** ■

COROLLARY 1: In these assumptions, **if R is Noetherian, then R^G is also Noetherian.**

Proof: Any infinite, strictly monotonous sequence $I_0 \subsetneq I_1 \subsetneq ...$ of ideals in R^G gives a strictly monotonous sequence $RI_0 \subsetneq RI_1 \subsetneq ...$ in R. ■

Graded rings

DEFINITION: A graded ring is a ring A^*, $A^* = \bigoplus_{i=0}^\infty A^i$, with multiplication which satisfies $A^i \cdot A^j \subset A^{i+j}$ ("grading is multiplicative"). A graded ring is called **of finite type** if all A^i are finitely dimensional.

We will usually assume that A^0 is the base field.

EXAMPLE: Polynomial ring $\mathbb{C}[V] = \bigoplus_i \operatorname{Sym}^i V$ is clearly graded.

Graded rings (2)

Claim 1: Let A^* be a graded ring of finite type. Then A^* **is Noetherian** $\Leftrightarrow$ **it is finitely generated**.

Proof. Step 1: If A^* is finitely generated, it is Noetherian by Hilbert's basis theorem.

Step 2: Conversely, suppose that A^* is Noetherian. Then the ideal $\bigoplus_{i>0} A^i \subset A^*$ is finitely generated. Let $a_i \in A^{n_i}$ be generators of this ideal over A^*. **We are going to show that products of a_i generate A^*.**

Step 3: Let $z \in A^*$ be a graded element of smallest degree which is not generated by products of a_i. Since a_i generate the ideal $\bigoplus_{i>0} A^i \subset A^*$, we can express z as $z = \sum_i f_i a_i$, where $f_i \in A^*$. However, $\deg f_i < \deg z$, hence all f_i are generated by products of a_i. Then all f_i are generated by products of a_i. ■

A caution: In this argument, two notions of "finitely generated" are present: finitely generated ideals (an additive notion) and finitely generated rings over $\mathbb{C}$ (multiplicative). **These two notions are completely different!** One is defined for ideals (or R-modules), another for a ring over a field. Only the name is the same (bad terminology).

Proof of Noether theorem for polynomial invariants

DEFINITION: Let V be a vector space with basis $z_1, ..., z_n$, and $\mathbb{C}[V] = \bigoplus_i \operatorname{Sym}^i V = \mathbb{C}[z_1, ..., z_n]$ the corresponding polynomial ring. Suppose that G acts on V by linear automorphisms. We extend this action to the symmetric tensors $\bigoplus_i \operatorname{Sym}^i V$ multiplicatively. This implies that G acts on $\mathbb{C}[V]$ by automorphisms. Such action is called **linear**.

CLAIM: (Noether theorem for polynomial invariants)
Let G act linearly on the polynomial ring $\mathbb{C}[V]$. **Then the invariant ring $\mathbb{C}[V]^G$ is finitely generated.**

Proof. Step 1: Since the action of G preserves the grading on $\mathbb{C}[V]$, **the ring $\mathbb{C}[V]^G$ is graded and of finite type.**

Step 2: $\mathbb{C}[V]^G$ is Noetherian, because $\mathbb{C}[V]$ is Noetherian, and the ring of invariants R^G is Noetherian if R is Noetherian (Corollary 1).

Step 3: A finite type Noetherian graded ring is finitely generated by Claim 1. ■

Noether theorem

THEOREM: (Noether theorem)
Let R be a finitely generated ring over $\mathbb{C}$, and G a finite group acting on R by automorphisms. Then **the ring R^G of G-invariants is finitely generated**.

Proof. Step 1: Let $f_1, ..., f_m$ be generators of R, and $\{g_1, ..., g_k\} = G$. Consider the space $V \subset R$ generated by all vectors $g_i f_j$. Clearly, $V \subset R$ is V-invariant, and **the natural homomorphism $\mathbb{C}[V] \longrightarrow R = \mathbb{C}[V]/I$ is surjective and G-invariant.**

Step 2: The natural map $\mathbb{C}[V]^G \longrightarrow R^G$ is surjective, because the functor $W \longrightarrow W^G$ is exact.

Step 3: The ring $\mathbb{C}[V]^G$ is finitely generated by Noether theorem for polynomial invariants, hence its quotient R^G is also finitely generated. ■

Tensor product of rings and preimage of a point

PROPOSITION: Let $f: X \longrightarrow Y$ be a morphism of affine varieties, $f^*: \mathcal{O}_Y \longrightarrow \mathcal{O}_X$ the corresponding ring homomorphism, $y \in Y$ a point, and $\mathfrak{m}_y$ its maximal ideal. **Denote by R_1 the quotient of $R := \mathcal{O}_X \otimes_{\mathcal{O}Y} (\mathcal{O}_Y/\mathfrak{m}_y)$ by its nilradical. Then $\operatorname{Spec}(R_1) = f^{-1}(y)$.**

Proof. Step 1: If $\alpha \in \mathcal{O}_Y$ vanishes in y, $f^*(\alpha)$ vanishes in all points of $f^{-1}(y)$. This implies that **the set V_I of common zeros of the ideal $I := \mathcal{O}_X \cdot f^*\mathfrak{m}_y$ contains $f^{-1}(y)$.**

Step 2: If $f(x) \neq y$, take a function $\beta \in \mathcal{O}_Y$ vanishing in y and non-zero in $f(x)$. Since $\varphi^*(\beta)(x) \neq 0$ and $\beta(y) = 0$, this gives $x \notin V_I$. **We proved that the set of common zeros of the ideal $I = \mathcal{O}_X \cdot f^*\mathfrak{m}_y$ is equal to $f^{-1}(y)$.**

Step 3: Now, strong Nullstellensatz implies that $\mathcal{O}_{f^{-1}(y)}$ is a quotient of $R = \mathcal{O}_X/I$ by nilradical. ■

EXERCISE: Give an example when $R = \mathcal{O}_X/I$ is non-reduced (contains nilpotents).

Finite quotients

CLAIM: Let R be a Noetherian ring without zero divisors, G a finite group acting by automorphisms on R, and R^G the ring of G-invariants. **Then** $\varphi: \operatorname{Spec} R \longrightarrow \operatorname{Spec} R^G$ **is a finite, dominant morphism**.

Proof. Step 1: For any $g \in G$, consider the corresponding polynomial map $P_g: R \longrightarrow R$, and let $r \in R$. The polynomial $P(t) := \prod_{g\in G}(t-g(r))$ has G-invariant coefficients for any $r \in R$, hence $P(t) \in R^G[t]$

Step 2: The morphism φ is finite because each $r \in R$ satisfies the equation $P(r) = 0$, where $P(t) = \prod_{g\in G}(t - g(r))$. It is dominant, because $R^G \subset R$. ∎

DEFINITION: Let G be a finite group acting on an affine variety X by automorphisms. **The quotient space** X/G is $\operatorname{Spec}(\mathcal{O}_X^G)$.

EXAMPLE: $\mathbb{C}^2/\{\pm 1\} = \mathbb{C}[x^2, y^2, xy] = \mathbb{C}[t_1, t_2, t_3]/(t_1 t_2 = t_3^2)$. Indeed, $\mathbb{C}^2/\{\pm 1\} = \operatorname{Spec} A$, where $A = \mathbb{C}[x, y]^{\{\pm 1\}}$: A **is the ring of even polynomials**.

EXAMPLE: Let $G = \mathbb{Z}/n\mathbb{Z}$ act on $\mathbb{C}$ by multiplication by a primitive root $\sqrt[n]{1}$. Then $\mathbb{C}/G = \operatorname{Spec}(\mathbb{C}[t]^G) = \operatorname{Spec}(\mathbb{C}[t^n])$, hence **the quotient space** $\mathbb{C}/G$ **is isomorphic to** $\mathbb{C}$.

Finite quotients (2)

THEOREM: Consider the natural morphism $\operatorname{Spec} R \xrightarrow{\varphi} \operatorname{Spec} R^G$. Then $\varphi(x) = \varphi(y)$ **if and only if** $x \in G \cdot y$, that is, the **set of points in** $\operatorname{Spec} R^G$ **is identified with the space of** G**-orbits**.

Proof. Step 1: If two maximal ideals of R are G-conjugated, their intersections with $R^G \subset R$ are equal. This gives $\varphi(gx) = \varphi(x)$: **each** G**-orbit is mapped to one point**. It remains to show that **the preimage of any point is exactly one** G**-orbit.**

Step 2: For any ideal $\mathfrak{m} \subset R^G$, one has $(\mathfrak{m}R)^G = \mathfrak{m}$. Then $A^G = R^G/\mathfrak{m}$, where $A := R \otimes_{R^G} (R^G/\mathfrak{m}) = R/\mathfrak{m}R$.

Step 3: Let $\mathfrak{m}$ be the maximal ideal of $y \in \operatorname{Spec} R^G$, and N the nilradical of $A := R/\mathfrak{m}R$. Since $\varphi^{-1}(y) = \operatorname{Spec}(A/N)$, **points of** $\varphi^{-1}(y)$ **are maximal ideals of the ring** A/N.

Step 4: A semisimple Artinian $\mathbb{C}$-algebra A/N is a direct sum of finite extensions of $\mathbb{C}$, which are all isomorphic to $\mathbb{C}$, giving $A/N = \bigoplus \mathbb{C}$. Since $A^G = \mathbb{C}$ (Step 2), **the group** G **acts on the summands of** $A/N = \bigoplus \mathbb{C}$ **transitively**. Therefore, **all points of** $\varphi^{-1}(y)$ **belong to the same** G**-orbit**. ∎