Complex analytic spaces

lecture 16: The Remmert rank

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IMPA, sala 236,

September 27, 2023, 17:00

Finite morphisms (reminder)

DEFINITION: Let $A \longrightarrow B$ be a homomorphism of rings. It is called **finite** if B is finitely generated as A-module.

EXAMPLE: $\mathbb{Z}[\sqrt{-1}\,]$ **is finite over** $\mathbb{Z}$**.**
EXAMPLE: (Finiteness theorem)
Let $z_1, ..., z_n$ be a regular coordinate system for an ideal $J \subset \mathcal{O}_n$, and $\mathcal{O}_d$ holomorphic functions, depending only on $z_1, ..., z_d$. Then **the ring $\mathcal{O}_n/J$ is finitely generated as a $\mathcal{O}_d$-module.**
Proof: Lecture 8. ■

In other words, **the ring morphism $\mathcal{O}_d \longrightarrow \mathcal{O}_n/J$ is finite.**

DEFINITION: Let Z, Z_1 be germs of complex varieties. Recall that **a morphism** $\varphi: Z \longrightarrow Z_1$ is a germ of a map which can is given in coordinates by holomorphic functions: if $Z \subset \mathbb{C}^n$ and $Z_1 \subset \mathbb{C}^m$ are germs of complex analytic subsets, **φ is expressed by an m-tuple of holomorphic functions $\varphi_1, ..., \varphi_m \in \mathcal{O}_n$ taking $z \in Z$ to $(\varphi_1(z), ..., \varphi_m(z)) \in Z_1$.**

DEFINITION: In this situation, the ring $\mathcal{O}_Z$ of germs of functions on Z is a $\mathcal{O}_{Z_1}$-module. We say that φ is **finite** if $\mathcal{O}_Z$ is finitely generated as a $\mathcal{O}_{Z_1}$-module, and **dominant** if $\varphi(Z)$ does not belong to a proper complex-analytic subvariety of Z_1.

Constant rank theorem (reminder)

DEFINITION: Let X be a complex manifold, and $F: X \longrightarrow \mathbb{C}^n$ a holomorphic map. **Rank of F in $x \in X$** is the number $\operatorname{rk}_x F := \dim X - \dim \ker dF$.

THEOREM: (Constant rank theorem)
Let $F: X \longrightarrow Y$ be a holomorphic map of complex varieties, where X is smooth. Assume that F has constant rank. **Then $F^{-1}(y)$ is a smooth submanifold of X for all $y \in F(X)$.** Moreover, **for any $x \in X$ there is a neighbourhood $U \ni x$ such that $F(U)$ is a smooth manifold of dimension $\operatorname{rk} F$.**

Proof. Step 1: If $\operatorname{rk} F = \dim X = n$ in $x \in X$, we compose F with a coordinate projection in order to obtain a map $F_1: X \longrightarrow \mathbb{C}^n$ with dF_1 invertible. **Then F_1 defines a local diffeomorphism of a neighbourhood $U \subset X$ of x to an open subset $W \subset \mathbb{C}^n$ by the inverse function theorem.** Therefore, $F(U)$, which is biholomorphic to W, is a smooth subvariety in Y.

Step 2: If $\operatorname{rk} F = k < \dim X = n$, we replace X by a sufficiently small neighbourhood of x, and take another map $G: X \longrightarrow \mathbb{C}^{n-k}$ such that dG is invertible on the space $\ker dF$ (which is $(n-k)$-dimensional). Then $F \times G: X \longrightarrow Y \times \mathbb{C}^{n-k}$ is a constant rank map which satisfies $\operatorname{rk} F \times G = \dim X$, and **Step 1 implies that it is biholomorphic to its image which is smooth and locally isomorphic to $\operatorname{im} F \times \mathbb{C}^{n-k}$.** Then $\operatorname{im} F$ is also smooth, and preimages of any $y \in F(Y)$ are smooth by the inverse function theorem. ∎

Maps with finite fibers (reminder)

We know that finite maps have finite fibers and preserve the dimension. Now we can prove that the finiteness assumption can be dropped: **all maps with finite fibers preserve the dimension.**

LEMMA: Let $F: X \longrightarrow Y$ be a holomorphic map of complex varieties. Assume that $F^{-1}(y)$ is finite for all $y \in Y$. **Then** $\dim X \leqslant \dim Y$**.**

Proof. Step 1: Using the finiteness theorem, we construct a finite map from Y to a disk U of the same dimension. This map has finite fibers. **Replacing** Y **by** U**, we may assume that** Y **is a disk in** $\mathbb{C}^n$.

Step 2: Let $x \in X$ be a point where dF has maximal rank k. Clearly, $\operatorname{rk} dF \geqslant k$ is equivalent to a non-vanishing of a $k \times k$ minor in the matrix dF. Locally in a neighbourhood of x, this minor is non-zero, hence $\operatorname{rk} dF = k$ in a neighbourhood U of x. Since the singular set is nowhere dense, we can always assume that U is smooth. **By the constant rank theorem,** $F(X)$ **is smooth and has the same dimension as** X**, which implies** $\dim X \leqslant \dim Y$**.** ■

Maps with finite fibers, part 2 (reminder)

Lemma 1: Let $Z \subset \mathbb{C}^n$ be an equidimensional subvariety, and $V \subset \mathbb{C}^n$ a k-dimensional subset. Assume that $\dim(V \cap Z) = 0$. **Then** $\dim Z \leqslant n - k$.
Proof: As shown in Lecture 14, $\dim Z_f = \dim Z - 1$ for any divizor $Z_f \subset Z$. Then induction gives $\dim(Z \cap V) \geqslant \dim Z - \operatorname{codim} V$. ■

Lemma 2: Let $Z \subset \mathbb{C}^n$ be an equidimensional subvariety of dimension d. Then for all $(n-d)$-dimensional affine subspaces V outside of a measure zero set of the relevant Grassmannian, **the intersection $Z \cap V$ has dimension 0.**
Proof. Step 1: Covering Z by a countable collection of open charts U_i, and proving the result for these charts, we obtain that the set of all V such that $Z \cap V$ has dimension 0 is a complement to a countable union of measure 0 sets. Therefore, it suffices to prove Lemma 1 locally for a sufficiently small open set. **We reduced Lemma 2 to a statement about a germ of a variety.**

Step 2: We assume that Z is a germ of a subvariety of $\mathbb{C}^n$ in 0. The set of linear coordinate frames is naturally identified with $GL(n,\mathbb{C})$. **Outside of a measure zero set in $GL(n,\mathbb{C})$, all linear coordinate systems $z_1, ..., z_n$ are regular.** Then the linear projection π_d to $\mathbb{C}^d$ is finite, and every point in $\mathbb{C}^d$ has finite preimage in Z. Outside of a measure 0 set, any $(n-d)$-dimensional subspace $V \subset \mathbb{C}^n$ can be realized as a preimage $\pi_d^{-1}(z)$ for an appropriate regular coordinate system, and $Z \cap V$ has dimension 0 for all such V. ■

Maps with finite fibers: semicontinuity (reminder)

(*) **PROPOSITION:** Let $F: \mathbb{C}^n \longrightarrow \mathbb{C}^m$ be a holomorphic submersion preserving 0, and $Z \subset \mathbb{C}^n$ a germ of a subvariety in 0, satisfying $F^{-1}(0) \cap Z = 0$. **Then F is proper in a neighbourhood of 0 and has finite fibers**.

Proof: Performing a local coordinate change, we may always assume that F is a linear projection. Let $U = D \times D'$ be a polydisc neighbourhood of $0 \in \mathbb{C}^n$, where F projects $D \times D'$ to D' along D. **Choosing D' sufficiently small, we can assume that $Z \cap \partial D \times D' = \emptyset$.** Indeed, $Z \cap \partial D \times D'$ is closed, and its intersection with $F^{-1}(0)$ is empty.

Step 2: Let $t \in D'$. Since $F^{-1}(t) \cap Z$ is a closed subset not intersecting the boundary, it is compact in $D \times \{t\}$. Then the restriction $F|_{Z \cap D \times D'}: Z \cap D \times D \longrightarrow D'$ is proper. **The fibers of F are compact closed subvarieties of a disk; they are finite by maximum principle**. ■

Remmert rank

DEFINITION: Let $x \in X$ be a point on a complex variety. **The dimension of X in x,** denoted $\dim(X,x)$, is maximum of dimensions for all irreducible components of X containing x; **this matters only when X is not equidimensional.**

DEFINITION: Let $F: X \longrightarrow Y$ be a holomorphic map of complex varieties. Define **the Remmert rank of F in $x \in X$** as $\operatorname{Rrk}_x F := \dim(X,x) - \dim(F^{-1}(F(x)),x)$.

REMARK: Clearly, $\operatorname{Rrk}_x F = \operatorname{rk}_x F$ **when x is smooth and F is a smooth submersion to its image.**

Remmert rank is semicontinuous

Theorem 1: The rank $\operatorname{Rrk}_x F$ is upper semicontinuous as a function of x (that is, the set of $x \in X$ where $\operatorname{Rrk} > t$ is open).

Proof. Step 1: Let $F: (X,x) \longrightarrow (Y,y)$ be a germ of a holomorphic map, and $\dim F^{-1}(y) = k$. We consider (X,x) as a germ of subvariety in $(\mathbb{C}^n, 0)$. Consider a general subspace $V \subset \mathbb{C}^n$ of dimension $n-k$; by Lemma 2, $V \cap F^{-1}(y)$ has dimension 0. By Proposition (*), the projection $F: (X,x) \cap V \longrightarrow (Y,y)$ has finite fibers in a certain neighbourhood of y. In other words, **$F^{-1}(F(x'))$ intersects V in a finite set for any x' in a sufficiently small neighbourhood of x.**

Step 2: By Lemma 1, $\dim(F^{-1}(F(x')) \cap V) = 0$ implies that $\dim F^{-1}(F(x') \leqslant k$. Therefore, $\dim F^{-1}(F(x')) \leqslant k = \dim(F^{-1}(F(x)), x)$. ■

REMARK: In fact, **Remmert rank is semicontinuous in analytic Zariski topology.** Indeed, the set of x' such that $F^{-1}(F(x')) \cap V$ is finite is the set of all x' such that the projection $F^{-1}(F(x')) \longrightarrow \mathbb{C}^n/V$ is not equal for π_d for a regular coordinate system. However, a coordinate system is regular when a collection of polynomials is Weierstrass, and **this is equivalent to non-vanishing of a certain homogeneous polynomial.**

Remmert rank theorem

THEOREM: Let $F: X \longrightarrow Y$ be a morphism of complex varieties, and $k := \sup_{x\in X} \operatorname{Rrk}_x F$. **Then $\operatorname{im} F$ belongs to a union of complex varieties of dimension $\leqslant k$,** but not in a union of complex varieties of dimension $\leqslant k-1$.

Proof. Step 1: Using induction, we may assume that this theorem is true for any map $F_1: X_1 \longrightarrow Y_1$ for which $\dim X_1 < \dim X$.

Step 2: The Remmert rank $\operatorname{Rrk}(F, x)$ is semicontinuous in x, hence it reaches its maximum on an open subset. The constant rank theorem implies that F is a smooth submersion on the set X_0 of smooth points of X where $\operatorname{rk} dF\big|_{T_xX}$ is maximal, and, moreover, **for a sufficiently small $U \subset X_0$, the image $F(U)$ is a complex submanifold of dimension k.**

Step 3: The complement $A := X\backslash X_0$ is complex analytic and has dimension $< \dim X$. By induction assumption, the theorem is true for A. The Remmert rank of $F|_A$ in $a \in A$ is

$$\dim(A,a) - \dim F^{-1}(F(a)) < \dim(X,a) - \dim F^{-1}(F(a)) \leqslant k = \sup_{x\in X} \operatorname{Rrk}_x F.$$

By induction assumption, $F(A)$ belongs to a union of subvarieties of dimension $\leqslant k$. ■

COROLLARY: (Remmert rank theorem:)
Let $F: X \longrightarrow Y$ be a holomorphic, surjective map of complex varieties. **Then** $\dim Y = \sup_{x\in X} \operatorname{Rrk}_x F$. ■