Complex analytic spaces

lecture 17: Remmert and Remmert-Stein theorems

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Remmert rank (reminder)

DEFINITION: Let $x \in X$ be a point on a complex variety. **The dimension of X in x,** denoted $\dim(X, x)$, is maximum of dimensions for all irreducible components of X containing x; **this matters only when X is not equidimensional.**

DEFINITION: Let $F: X \longrightarrow Y$ be a holomorphic map of complex varieties. Define **the Remmert rank of F in $x \in X$** as $\operatorname{Rrk}_x F := \dim(X, x) - \dim(F^{-1}(F(x)), x)$.

REMARK: Clearly, $\operatorname{Rrk}_x F = \operatorname{rk}_x F$ **when x is smooth and F is a smooth submersion to its image.**

Theorem 1: The rank $\operatorname{Rrk}_x F$ is upper semicontinuous as a function of x (that is, the set of $x \in X$ where $\operatorname{Rrk} > t$ is open).

Remmert rank theorem (reminder)

THEOREM: Let $F: X \longrightarrow Y$ be a morphism of complex varieties, and $k := \sup_{x\in X} \operatorname{Rrk}_x F$. **Then $\operatorname{im} F$ belongs to a union of complex varieties of dimension $\leqslant k$,** but not in a union of complex varieties of dimension $\leqslant k-1$.

Proof. Step 1: Using induction, we may assume that this theorem is true for any map $F_1: X_1 \longrightarrow Y_1$ for which $\dim X_1 < \dim X$.

Step 2: The Remmert rank $\operatorname{Rrk}(F,x)$ is semicontinuous in x, hence it reaches its maximum on an open subset. The constant rank theorem implies that F is a smooth submersion on the set X_0 of smooth points of X where $\operatorname{rk} dF\big|_{T_xX}$ is maximal, and, moreover, **for a sufficiently small $U \subset X_0$, the image $F(U)$ is a complex submanifold of dimension k.**

Step 3: The complement $A := X\backslash X_0$ is complex analytic and has dimension $< \dim X$. By induction assumption, the theorem is true for A. The Remmert rank of $F|_A$ in $a \in A$ is

$$\dim(A,a) - \dim F^{-1}(F(a)) < \dim(X,a) - \dim F^{-1}(F(a)) \leqslant k = \sup_{x\in X} \operatorname{Rrk}_x F.$$

By induction assumption, $F(A)$ belongs to a union of subvarieties of dimension $\leqslant k$. ■

COROLLARY: (Remmert rank theorem:)
Let $F: X \longrightarrow Y$ be a holomorphic, surjective map of complex varieties. **Then** $\dim Y = \sup_{x\in X} \operatorname{Rrk}_x F$. ■

Reinhold Remmert (22 June 1930 - 9 March 2016)
August 1983, Oberwolfach, photo by Paul Halmos.

Remmert and Remmert-Stein theorem

DEFINITION: A proper map is a continuous map such that a preimage of a compact is always compact.

THEOREM: (Remmert-Stein theorem)
Let X be a complex variety, $A \subset X$ a complex analytic subset, and Z an irreducible complex analytic subset in $X \backslash A$. Assume that $\dim Z > \dim A$. **Then the closure of Z is complex analytic in X.**

EXERCISE: Deduce the Riemann removable singularity theorem from Remmert-Stein, in the following form. **Let f be a continuous function on $\mathbb{C}$, holomorphic outside of a discrete set. Prove that f is holomorphic.**

THEOREM: (Remmert proper map theorem)
Let $F: X \longrightarrow Y$ be a proper morphism of complex varieties. **Then $F(X)$ is complex analytic in Y.**

Karl Stein (1913-2000)
Eichstätt, 1968

Bounded holomorphic functions

Extension lemma: Let $D \subset B$ be a divisor in a ball $B \subset \mathbb{C}^n$, and $f \in \mathcal{O}_{B\backslash D}$ a bounded holomorphic function on its complement. **Then f can be extended to a holomorphic function on B.**

Proof. Step 1: Let u be a holomorphic functions vanishing on D. Then fu^2 vanishes on D, and its derivative also vanishes on D, **hence fu^2 is holomorphic on B.**

Step 2: This implies that f is meromorphic, $f = \frac{g}{h}$. **Using factoriality of the ring of germs, we may assume that g and h are coprime, and h is not divisible by the prime factors of g, denoted as g_i.** It remains to show that h is invertible. On contrary, assume it is not invertible.

Step 3: Since h is coprime with g, it does not belong to the ideal (g_i). Its zero divisor D_h has dimension $(n-1)$, hence it cannot be properly contained in a union of irreducible divisors D_{g_i}. **This implies that there is a point $x \in B$ where h vanishes and g does not vanish.**

Step 3: In a neighbourhood of x, the fraction $f = \frac{g}{h}$ is not bounded, which brings a contradiction. ■

REMARK: This statement is also true when $D = A \cup Z$, where $A \subset B$ is a divisor, and $Z \subset B\backslash A$ a divisor in $B\backslash A$. Indeed, in this situation f can be holomorphically extended to $B\backslash A$, because Z is a divisor in $B\backslash A$, and then on B, because $A \subset B$ is a divisor. **We will often use the extension lemma in this form.**

Extension lemma for coverings

REMARK: Let G be a finite group acting on $\mathbb{C}^n$ holomorphically. **The quotient $X := \mathbb{C}^n/G$ is a complex variety,** with $\mathcal{O}_X$ being G-invariant holomorphic functions on $\mathbb{C}^n$. We shall prove this result later.

DEFINITION: Let B be an open ball. **The symmetric power** $\operatorname{Sym}^q B$ is a complex variety (generally speaking, singular). Its points can be encoded by unordered q-tuples $\alpha_1, ..., \alpha_q \in B$. The coordinates on this variety are given by certain symmetric polynomials on the coordinates of α_i.

Lemma 1: Let B_1, B_2 are open balls, $A \subset B_1$ a union of a divisor $D \subset B$ and a divisor $D' \subset B \backslash D$, and $Z \subset (B_1 \backslash A) \times B_2$ a closed submanifold, which is projected to B_2 as a q-sheeted unramified finite covering. **Then the closure of Z in $B_1 \times B_2$ is complex analytic.**

Proof: We interpret Z as a graph of a multivalued holomorphic function $\varphi: B_1 \backslash A \longrightarrow \operatorname{Sym}^q B_2$. Locally this map can be expressed as an unordered collection of maps $\zeta_1, ..., \zeta_q : B_1 \backslash A \longrightarrow B_2$. The coefficients of symmetric polynomials of the coordinates of $\zeta_1, ..., \zeta_q$ are holomorphic functions on $B_1 \backslash A$ which are bounded, hence, by extension lemma, can be extended to holomorphic functions on B_1. Therefore, $\varphi : B_1 \backslash A \longrightarrow \operatorname{Sym}^q B_2$ can be extended to a holomorphic function $\tilde{\varphi} : B_1 \longrightarrow \operatorname{Sym}^q B_2$. **The closure of Z is the graph of the corresponding multivalued function.** ■

Remmert and Remmert-Stein theorem (scheme of the proof)

The proof follows from an elaborate induction argument.

(RS_m): the statement of Remmert-Stein is true when $\dim X \leqslant m$.
(R_m): the statement of Remmert proper map is true when $\dim X \leqslant m$.

Today we prove two statements, which together bring forth the inductive argument.

A. (RS_m) and (R_{m-1}) implies (R_m).
B. (R_{m-1}) implies (RS_m).

Remmert and Remmert-Stein theorem: implication A.

THEOREM: (Remmert proper map theorem)
Let $F: X \longrightarrow Y$ be a proper morphism of complex varieties. **Then $F(X)$ is complex analytic in Y.**

(RS$_m$): the statement of Remmert-Stein is true when $\dim X \leqslant m$.
(R$_m$): the statement of Remmert proper map is true when $\dim X \leqslant m$.

Proof of implication A: (RS$_m$) and (R$_{m-1}$) implies (R$_m$).

Step 1: Let $X_1 \subset X$ the set of all points where the Remmert rank $\operatorname{Rrk}_x(F)$ is not maximal. By constant rank theorem, **$F(X)$ is complex analytic in a neighbourhood of any point $y \in Y$ outside of $F(X_1) \cup F(X_{\text{sing}})$.** Indeed, the preimage of y is compact, and covered by a finite collection of open sets where the constant rank theorem holds.

Step 2: Using R$_{m-1}$, we obtain that **the images $F(X_1)$ and $F(X_{\text{sing}})$ are complex-analytic**. By Remmert rank theorem, its image has dimension $\leqslant$ $\max_x \operatorname{Rrk}_x F\Big|_{(X_1 \cup X_{\text{sing}})}$, and this inequality is never strict.

Remmert and Remmert-Stein theorem: implication A (2).

Proof of implication A: (RS$_m$) and (R$_{m-1}$) implies (R$_m$).

Step 1: Let $X_1 \subset X$ the set of all points where the Remmert rank $\operatorname{Rrk}_x(F)$ is not maximal. Then $F(X)$ **is complex analytic in a neighbourhood of any point** $y \in Y$ **outside of** $F(X_1) \cup F(X_{\mathsf{sing}})$.

Step 2: Using R$_{m-1}$, we can assume that **the images** $F(X_1)$ **and** $F(X_{\mathsf{sing}})$ **are complex-analytic**.

Step 3: Let $X' := X \backslash F^{-1}(F(X_1 \cup X_{\mathsf{sing}}))$. Remmert rank theorem gives

$$\dim F(X_{\mathsf{sing}}) = \sup_{x \in X_{\mathsf{sing}}} \operatorname{rk}\left(F\Big|_{X_{\mathsf{sing}}}, x\right) =$$

$$\dim X_{\mathsf{sing}} - \inf_{x \in X_{\mathsf{sing}}} \dim F^{-1}(F(x)) < \operatorname{rk} \sup_{x \in X} \operatorname{rk}(F, x) = \dim F(X').$$

Similarly, $\dim F(X_1) = \sup_{x \in X_1} \operatorname{rk}\left(F\Big|_{X_1}, x\right) < \sup_{x \in X} \operatorname{rk}(F, x) = \dim F(X')$.

Step 4: Now **R$_m$ is implied by applying RS$_m$ to** $Z = F(X')$ **and** $A = F(X_1) \cup F(X_{\mathsf{sing}})$. ∎

Remmert and Remmert-Stein theorem: implication B.

THEOREM: (Remmert-Stein theorem)
Let X be a complex variety, $A \subset X$ a complex analytic subset, and Z an irreducible complex analytic subset in $X \backslash A$. Assume that $\dim Z > \dim A$. **Then the closure of Z is complex analytic in X.**
($\mathbf{RS}_m$): the statement of Remmert-Stein is true when $\dim X \leqslant m$.
($\mathbf{R}_m$): the statement of Remmert proper map is true when $\dim X \leqslant m$.

Proof of implication B: ($\mathbf{R}_{m-1}$) implies ($\mathbf{RS}_m$).
Step 1: Using induction in $\dim A$, **we may assume that A is smooth.** We can also assume that **Z is irreducible:** indeed, the intersection of irreducible components of Z has dimension $< \dim Z$.

Step 2: Applying an appropriate holomorphic diffeomorphism and passing to a neighbourhood of a given point $a \in A$, we may assume that X is a subvariety of a ball $B \subset \mathbb{C}^n$, and $A \subset B$ is a linear subspace. Consider a linear function on B, which is not identically zero on Z. Its intersection with Z is a divisor. Using induction, **we construct a linear projection $F: B \longrightarrow \mathbb{C}^{\dim Z}$ such that $F^{-1}(0) \cap (Z \cup A)$ is countable.**

Remmert and Remmert-Stein theorem: implication B (2).

Step 2: Applying an appropriate holomorphic diffeomorphism and passing to a neighbourhood of a given point $a \in A$, we may assume that X is a subvariety of a ball $B \subset \mathbb{C}^n$, and $A \subset B$ is a linear subspace. Consider a linear function on B, which is not identically zero on Z. Its intersection with Z is a divisor. Using induction, **we construct a linear projection $F: B \longrightarrow \mathbb{C}^{\dim Z}$ such that $F^{-1}(0) \cap (Z \cup A)$ is 0-dimensional.**

Step 3: Choose a polydisk $D \times D' \subset B$ such that **the projection F maps $D \times D'$ to D', and its restriction to $(Z \cup A) \cap D \times D'$ is proper and has countable fibers.** By taking a linear projection generic enough, we can always make sure that its restriction to A is finite, and its restriction to Z has 0-dimensional fibers. Then for all $t \in D' \backslash F(A)$ the intersection $F^{-1}(t) \cap (Z)$ is 0-dimensional. Choose the radius of D such that the intersection $F^{-1}(t) \cap (Z \cup A)$ is empty. Since the the distance from $(Z \cup A)$ to the boundary of $F^{-1}(0)$ is bigger than $\varepsilon > 0$, for all t in a sufficiently small neighbourhood of 0, the distance from $(Z \cup A)$ to the boundary of $F^{-1}(0)$ is bigger than $\frac{1}{2}\varepsilon$. Replacing D' by this neighbourhood, we make sure that $(Z \cup A) \cap \partial D \times D' = \emptyset$. Since $F^{-1}(t) \cap (Z)$ is 0-dimensional complex analytic for all $t \in D' \backslash F(A)$ and belongs to a compact subset of an open ball, this intersection is finite.

Remmert and Remmert-Stein theorem: implication B (3).

At this point we replace Z, A, X by their intersection with $D \times D'$.

Step 4: Let A' be the union of A and the set of all points where the differential of $F|_Z$ is not an isomorphism, and $Z' := Z \backslash F^{-1}(F(A'))$. Then the map $F|_{Z'}$ is a finite covering. The intersection $A' \cap Z$ is the set of all $z \in Z$ where $dF|_Z$ is not of maximal rank, hence it is complex analytic in Z, of dimension $\leqslant m-1$. **By $\mathbf{R}_{m-1}$, the image $F(A' \cap Z)$ is complex analytic.** The same is true for $F(A)$, because A is a vector space and F is linear. We obtain that $F(A')$ is a union of a complex subvariety and a linear subspace. **This is a situation when one could apply the extension lemma.**

Step 5: The subvariety $Z' \subset D \times D' \backslash F^{-1}(F(A'))$ is a graph of a q-sheeted covering $D' \backslash F(A') \longrightarrow \operatorname{Sym}^q(D)$. By Lemma 1 above, its closure in $D \times D'$ is complex analytic. ■